PB-229 918

UNSTEADY AND UNSATURATED FLOW IN SOILS IN TWO DIMENSIONS

Rameshwar Singh

Stanford University
Stanford, California

July 1965

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

PB 229 918

July 1965
Technical Report No. 54

UNSTEADY AND UNSATURATED FLOW IN SOILS IN TWO DIMENSIONS

RAMESHWAR SINGH

PREPARED UNDER
U.S. PUBLIC HEALTH SERVICE
RESEARCH GRANT WP-00246

Department of CIVIL ENGINEERING
STANFORD UNIVERSITY

HD

DEPARTMENT OF CIVIL ENGINEERING

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

UNSTEADY AND UNSATURATED FLOW IN SOILS IN TWO DIMENSIONS

By

Rameshwar Singh

Ph.D.

Technical Report No. 54

July 1965

This research received support from the U. S. Public Health Service through Research Grant WP-00246.

ACKNOWLEDGMENTS

The writer wants to express deep and sincere appreciation to Professor Joseph B. Franzini, acting executive head of the Civil Engineering Department for his kind help during the writer's entire study period at Stanford University. Professor Franzini's counsel was always available in guiding the writer through all difficulties whether personal or academic. His valuable suggestions during the course of this research have contributed much to its success. Thanks and appreciation are also extended to Professor Norman H. Crawford and Professor Stanley N. Davis for their timely advice and critical review of the manuscript. Thanks must also go to Professor Melville S. Priest, the writer's M.S. thesis adviser at Auburn University, for his encouragement and help in persuading the writer to come to Stanford for the doctoral studies. The writer also wants to express his thanks to his father and wife for their good wishes for the success of the study. Finally, appreciation is extended to the U. S. Public Health Service for financial support under Research Grant WP-00246, which made this study possible.

ABSTRACT

The equation for unsteady and unsaturated flow of water in soils was derived by Klute in 1952 and since then many efforts have been made to obtain a solution to this equation. Some successful numerical procedures for the solution have been developed for the one-dimensional form of the equation, but the two- and three-dimensional forms of this equation have been untouched because of the mathematical complexities involved.

The writer has succeeded in obtaining a solution to the two-dimensional diffusion equation. The physical problem that is solved is that of flow from an underground cylindrical source of finite size. The equation is first transformed from cartesian coordinates to polar coordinates. After neglecting gravity, the equation is reduced to a simpler form which can be transformed to a nonlinear ordinary differential equation. This transformation is achieved after substituting two empirical relations for two parameters present in the diffusion equation, namely the hydraulic conductivity and the diffusivity. This ordinary differential equation is then solved by numerical techniques. The resulting solution is that for two-dimensional flow from the source without gravity.

To account for the gravity, an approximate correction based on the consideration of the diffusion equation in difference notation is applied. Two soils, Yolo Light Clay and Pachappa Loam, have been used for examples. Infiltration rates have been computed in each case; these are found to be almost constant for this problem. A computer model in extended Algol

language for the Burroughs B 5500 machine has been developed, encompassing all these methods, and which can be used for any soil having an exponential diffusivity relationship. The model is very fast on the computer and accuracy can be varied as desired. Anybody with some introduction to Algol can easily use this model without knowing the detail of all of the mathematical treatments. This solution can be applied to underground porous or tile pipes diffusing water out into the soil. It can also be applied to many other problems such as the disposal of diluted nuclear wastes in deep wells.

Some work done by the writer on the three-dimensional diffusion equation is presented in Appendix 2. These equations can probably be solved by similar techniques. This work opens up many avenues for further research, a few of which are mentioned in Chapter 9.

TABLE OF CONTENTS

Chapter		Page
1	Introduction	1
2	History and General Introduction to Subsurface Flow	5
	2.1 Historical Sketch of General Theories	5
	2.2 Background of Unsaturated Flow	7
	2.3 Assumptions in Unsaturated Flow	14
3	Hydraulic Conductivity and Diffusivity of Unsaturated Soils	21
4	Transformation of Diffusion Equation into Polar Coordinates	33
5	Solution of the Difference Equation by Implicit Difference Method	41
6	Transformation of the Diffusion Equation to an Ordinary Differential Equation	47
7	Approach to the Solution of the Diffusion Equation	63
	The Runge-Kutta Method	63
	Adams Method	69
	Method of Solution	73
	Method of Accounting for Gravity	78
	Computation of Infiltration Rate	81
8	Operation of Computer Program and Examples	84
9	Conclusion	106
Appendices		110
	Appendix 1: A. Algol Program	110
	B. Flow Chart for Algol Program	113
	Appendix 2: Three Dimensional Diffusion Equation	114
Bibliography		124

LIST OF TABLES

		Page
8.1	Changes in the Program	85
8.2	Terms in Output	91
8.3	Solution without Gravity for r_o = 1 cm., Yolo Light Clay	92
8.4	Final Result at 45° Intervals, Moisture Content with Gravity	94
8.5	Solution without Gravity for r_o = 1 cm., Pachappa Loam	96
8.6	Final Result at 45° Intervals, Moisture Content with Gravity	98
8.7	Variation of Infiltration Rate with Radius of the Source	105

LIST OF FIGURES

Figure		Page
2.1	Diagram of Soil Column Standing in Water	9
2.2	Energy Relations for Greenville Loam	11
2.3	Schematic Representation of Moisture Profile	12
2.4	Typical Relationship between Volume Changes and Alternate Wetting and Drying of Soils	17
3.1	Schematic Representation of Relation between Threshold Moisture Content θ_o and Inactive Moisture Content θ_i	23
3.2	Relation between Hydraulic Conductivity K and Moisture Content θ for Yolo Light Clay	25
3.3	Relation between Diffusivity D and Moisture Content θ for Yolo Light Clay	28
3.4	Soil Moisture Diffusivity as a Function of Percent Moisture for Three Soils	30
3.5	Relationship between Hydraulic Conductivity and Capillary Head	32
4.1	Relation between Cartesian and Polar Coordinates	34
5.1	Grid for Difference Equation	42
6.1	Effect of Gravity	48
6.2	y-curves with Different Initial Values of y'	57
7.1	Step by Step Integration of 1st Order Ordinary Differential Equation	64
7.2	Integration of Differential Equation	74
7.3	Solution with Guessed $y'(x_o)$	76
7.4	Interpolation for $y'(x_o)$	76
7.5	Computation of Infiltration	82

Figure		Page
8.1	Moisture Profile at Different Times for Yolo Light Clay	101
8.2	Infiltration Curve for Yolo Light Clay	102
8.3	Moisture Profile for Pachappa Loam	103
8.4	Infiltration Rate for Pachappa Loam	104
A2.1	Relation between Cartesian and Spherical Coordinates	115

CHAPTER I

INTRODUCTION

Mankind has been making efforts to procure and utilize underground water since ancient times, but the mechanics of underground flow has been reasonably well understood during only the last 100 years. The discovery of the law of fluid flow through porous media by Henry Darcy, a French engineer, in 1856, was the beginning of our understanding of the phenomenon and led to the solution of varieties of subsurface flow problems. This law is justly known as Darcy's law. By the use of Darcy's law, many problems of the flow of fluids through saturated soils have been solved in the past, but the success in the solution of unsteady and unsaturated flows in soils has been limited because of mathematical complexities.

"Unsaturated flow" in soils is defined as flow in which all pore spaces are not completely filled with flowing liquid. Some authors prefer calling it "partially saturated flow" implying the same meaning. In this paper, both names will be used interchangeably.

In nature, there are many situations where steady and saturated flow occurs, such as flow in underground aquifers. In other instances the flow is saturated and unsteady. Solutions for many such problems have been worked out and are available in such references at Todd (1959) and Harr (1962). In recent years, the importance of unsteady and unsaturated flow has been recognized. The most obvious example of unsteady and unsaturated flow is the infiltration of water through the ground surface. Actually unsaturated flow through porous media is significant

to many fields of science and engineering. A hydrologist wants to know the rate of infiltration of rainwater through the ground surface, so that he may be able to evaluate the surface runoff which is the difference between precipitation and infiltration. After rain has ceased, the downward movement of water in soils presents a complicated case of partially saturated flow. The capillary fringe of an unconfined aquifer has a varying degree of saturation, and the flow through it might be a substantial portion of the total flow through a thin and sloping aquifer. Problems of draining soils are encountered by specialists in several branches of science; in such problems partially saturated flow takes place. If the ground water level is close to the surface, evaporation may remove water from the aquifer through unsaturated flow. This problem was treated by Gardner (1957). In cases of ground water level fluctuations, the phenomenon of unsaturated flow if often involved.

The rate of intake of water by soils in irrigation is important to agriculturalists, agronomists, civil engineers and others. The understanding and solution of problems of unsteady and unsaturated flow will help in the design of furrow channels and distribution systems. Losses of water from canals, ditches and leaky pipes can in some instances be estimated correctly only by unsaturated techniques.

The underground disposal of sewage and waste water is used in many places. Septic tanks and underground drains are used worldwide. In these circumstances unsaturated flow phenomena occur and analysis of such flows is sometimes required to determine the possibility of contamination of reservoirs of potable and usable water by the disposal sources. Another source of pollution is the underground disposal of

nuclear wastes, which requires particular care because of the high degree of hazard involved. To determine safe locations of liquid waste disposal sites and maximum rates of disposal, it is essential to understand fully unsaturated flow phenomena. Petroleum engineers also encounter problems of partially saturated flow, when extracting petroleum from the earth.

Though the importance of unsteady and unsaturated flow need not be emphasized further, it should be pointed out that the field situations that are encountered are not easily described mathematically. Soil structure is heterogeneous with varying porosity, particle size, orientation and gradation resulting in pores of various size and shape. The pores are partially filled with air which affects the flow of water. The mathematical formulation of a problem, taking into account all characteristics of the soil is difficult, and has not yet been accomplished. Even in the idealized case of homogeneous and airfree soils, the partial differential equation describing the flow is badly nonlinear and evidently unsteady, the analytic solution of which even in the simple case of one dimensional flow is not yet available.

In some special and simple cases, actual field conditions of unsaturated flow can be approximated by model studies in the laboratory, but generally it is difficult to duplicate the partially saturated flow phenomenon encountered in the field. Samples taken from the field for investigation may depart substantially from reality during the process of procurement and transfer. It is impossible to take enough field samples to get a completely accurate picture of the field conditions.

Numerical techniques have been used by many investigators for the solution of one dimensional problems. To the knowledge of this writer,

a solution is not yet available for the general case of unsteady and unsaturated flow in two and three dimensions. Nonlinearity and poor convergence characteristics of the equation is the reason for so little progress in the solution of this problem. This writer has endeavored to solve the two dimensional diffusion equation in polar co-ordinates first without gravity and then has applied a correction to account for the gravity. It is hoped that the result will be useful for many applications and also for future researches in this field.

CHAPTER 2

HISTORY AND GENERAL INTRODUCTION TO SUBSURFACE FLOW

2.1 Historical sketch of general theories.

The vital role of water in the existence of mankind on the earth is the chief reason for man's interest in groundwater. Long before recorded history, primitive man had to look for a source of water to quench his thirst. During dry periods, when surface sources of water were dry and not within easy reach of primitive communities, a search was made for some other source of this vital liquid. It was perhaps first found at shallow depths in dry beds of nearby rivers. Eventually, man sought water at greater depths and today it is not uncommon to get water from several hundred feet below the earth's surface.

Many descriptions of water wells are found in the ancient literature of the world. The Old Testament mentions water wells in Egypt. This writer knows of a large well that is nearly 2000 years old and still in use in his home state of India. This well is known as "Agam" well, constructed by King Ashoka at Patna, Bihar, India. Similarly, water was drawn from wells in ancient China, Greece and the Roman Empire.

Though groundwater was used by ancient people, little was understood about the hydrologic cycle and the mechanism of subsurface flow. Many writers and philosophers of those periods tried to hypothesize on the supply of water in wells and springs and had theories about precipitation, but most of the theories were very misleading and far from reality. The knowledge of geology of the earth's surface was lacking at the time. The first actual knowledge in this field in recorded

history is found to be that of the Roman architect Vitruvius, who conceived of the infiltration of rain water through the ground surface. After that, no advancement was made in the understanding of the hydrologic cycle till the end of the 17th Century. The most plausible reason seems to be the lack of experimental techniques and actual observations.

By the 17th Century, the fundamentals of the physical sciences were established and the importance of experiments to substantiate theories was also better realized. In the latter part of the 17th Century, many scientists measured and observed precipitation and stream flow. In this way a better grasp of the hydrologic cycle was obtained. The most notable contributions were those of the Frenchman Pierre Perrault, the Englishman Edmund Halley and the French physicist Edmé Mariotte.

During the 18th Century, more knowledge about geology and the earth's structure was acquired. Filtration of water through sandbeds for purification gave insight into the movement of water through soils. Then the light came in the mechanics of groundwater flow through the findings of Henry Darcy (1803-1858), a French engineer. His result was obtained experimentally and it proved to be the most vital tool in the solution of problems of the movement of all fluids through granular materials. Darcy's law states that

$$q = - K \frac{\partial h}{\partial \ell} \qquad (2.1)$$

where the terms are defined as follows:

q: discharge per unit area of the porous medium

K: hydraulic conductivity

(depends upon fluid, soil saturation and soil properties)

$\frac{\partial h}{\partial \ell}$: hydraulic gradient

The negative sign is used on right side of equation (2.1) because the head decreases in the direction of flow. Some people prefer using the positive sign instead for simplicity of expression.

After Darcy's discovery, various kinds of flow problems related to fully saturated media were solved. With the help of Darcy's law, many models of groundwater flow have been constructed. Graphical and numerical techniques have been devised, which are used extensively by civil engineers, petroleum engineers, agriculturists, geologists and others. Some of the people who contributed significantly to the solutions of important groundwater flow problems related to fully saturated media are J. Boussinesq, G. A. Daubrée, J. Dupuit, A. Theim, A. Hazen and O. E. Meinzer.

2.2 Background on unsaturated flow.

The discovery of unsaturated flow mechanics is very recent. To understand the phenomenon of unsaturated flow through porous media, it is necessary to understand various forces acting to cause the flow. Capillary suction is the most important force in this regard, though gravity also plays a role particularly in vertical unsaturated flow. Capillary phenomenon of soil water was investigated and explained by Buckingham (1907). In this context, a short description of this phenomenon is given below.

The capillary rise of water in a small glass tube is well known. Water rises in the tube and attains its equilibrium at a height where it forms a hemispherical meniscus. The thinner the tube the higher is the rise of water in the tube. This phenomenon is caused by the surface tension of water and greater molecular attraction between glass and water than between air and water. Water has similar affinity for rocks, sand grains, and soil particles. The rise of water in soil columns in the laboratory and in capillary fringes above aquifers are analogous phenomena.

Partially saturated flow in soil takes place because water is pulled from saturated or more saturated regions of the soil toward the dry or less saturated regions by the capillary suction. To express this phenomenon in physical terms, use is made of the concept of capillary potential ψ introduced by Buckingham. This concept is analogous to the concept of flow of electricity under the influence of a driving electromotive force, known as electrical potential. The work required to lift a unit mass of water from the water surface into a soil column is called the capillary potential. A negative sign is given to this work because of the upward movement of water against the action of gravity.

Capillary potential is a function of moisture content θ and is expressed as follows:

$$\frac{\partial \psi}{\partial \theta} = g \frac{\partial y}{\partial \theta} \qquad (2.2)$$

where y: vertical distance upwards from the water surface as shown in Fig. 2.1

g: acceleration due to gravity

With the boundary condition $\psi = 0$ at $y = 0$, Eq. 2.2 reduces to

$$\psi = gy \tag{2.3}$$

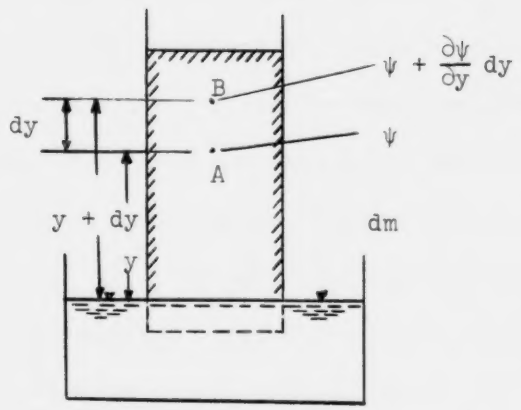


Fig. 2.1: Diagram of Soil Column Standing in Water.

The capillary potential is equal to the hydrostatic pressure of the soil water in metric units. Instead of using the term "capillary potential" with a negative sign, the term "soil moisture tension" is usually emplcyed with a positive sign, since in pulling water into soil interstices, a kind of tensile or attracting force acts. Its value

is generally expressed in terms of height of a liquid column needed to produce it.

Buckingham tested many soil columns to determine the relationship between moisture content and soil moisture tension, and found a continuous relationship between them. He also concluded that with the same value of soil moisture tension, a finer soil has a bigger value of moisture content than a coarser soil. Some people have suggested the use of semilogarithmic paper for plotting these relations to minimize difficulties due to the extreme range of the potential.

An unfortunate aspect of the relationship between soil moisture tension and soil moisture is that it is not constant at all times. Schofield (1935) has shown that the tension is higher for the same soil at the same moisture content during drying of the soil than during wetting as shown in Fig. 2.2. The term "pF" on the plot is the common logarithm of soil moisture tension. One must be very careful in the use of such relationships due to this hysterisis characteristic of soil moisture as indicated on Fig. 2.2.

Following Buckingham there was not much activity in the development of the science of unsaturated flow in soils until Richards (1931), Childs and George (1950) and others extended the idea of capillary potential to unsaturated flow and studied the concept of diffusivity. Richards, as a rough approximation, applied a constant diffusivity to problems of unsaturated flow.

During the same period, an important experiment was conducted by Bodman and Colman (1944) to determine the moisture distribution profile during the downward movement of water into soils. They used

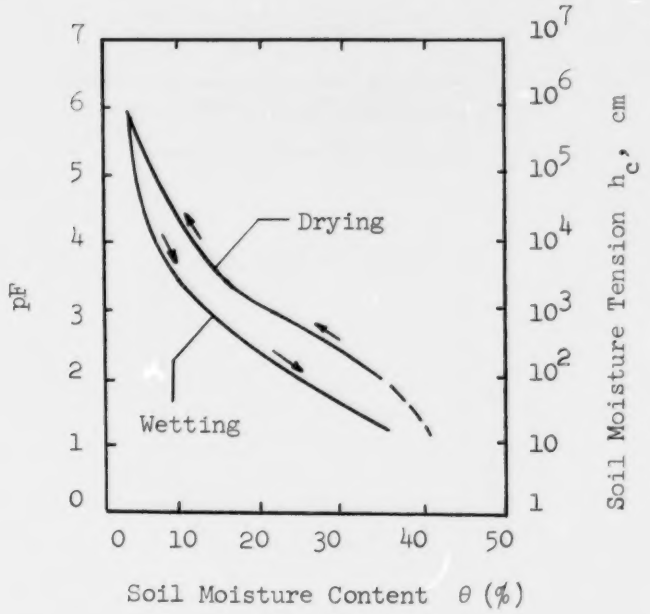


Fig. 2.2: Energy Relations for Greenville Loam (After Schofield 1935).

two kinds of soil and arrived at moisture profiles which were similar in shape for both soils, but the wetting fronts moved at different velocities. For the sake of clarity, they divided the profile into five regions depending upon uniformity and distribution of moisture content in each of its regions as shown in Fig. 2.3. Recent experiments and numerical solutions by many investigators do not show the existence of the transition zone.

Finally, A. Klute (1952) worked with the general equation of unsteady and unsaturated flow in soils in three-dimensions in vector

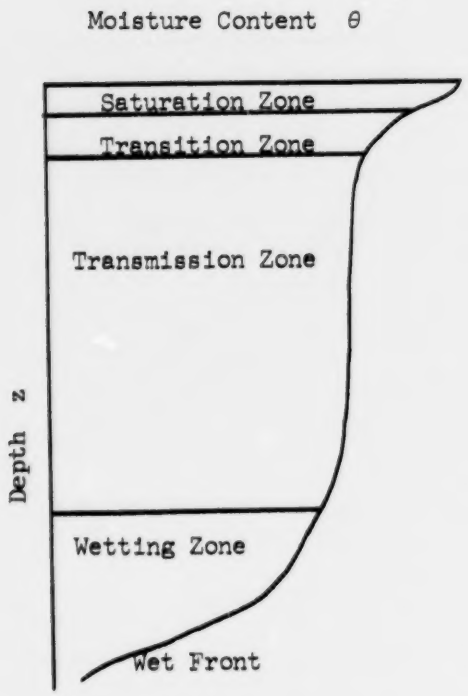


Fig. 2.3: Schematic Representation of Moisture Profile (After Bodman and Colman 1944).

notation. In this equation the concept of soil moisture tension is combined with continuity. Many efforts have been made to solve this equation. It is nonlinear and unsteady, and an analytic solution has not been achieved. In numerical solutions it has poor convergence characteristics.

Klute's equation in vector notation is

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (-K \nabla \Phi) \qquad (2.4)$$

where Φ: total potential per unit weight of fluid

t: time

Equation (2.4) can be written in ordinary notation as

$$\frac{\partial \theta}{\partial t} = -\left[\frac{\partial}{\partial x}\left(K \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial \Phi}{\partial z}\right)\right] \qquad (2.5)$$

where $\Phi = h_c + z$ and where h_c is the capillary head (capillary potential per unit weight) and z is the elevation of the fluid element above the reference datum. Then Eq. (2.5) becomes

$$\frac{\partial \theta}{\partial t} = -\left[\frac{\partial}{\partial x}\left(K \frac{\partial h_c}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial h_c}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial h_c}{\partial z}\right) + \frac{\partial K}{\partial z}\right] \qquad (2.6)$$

Defining diffusivity as $D(\theta) = -\frac{\partial h_c}{\partial \theta} \cdot K(\theta)$ and substituting in Eq. (2.6), the final result in three dimensions becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta) \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(D(\theta) \frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(D(\theta) \frac{\partial \theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z} \qquad (2.7)$$

Some successful numerical solutions have been obtained for the one dimensional case of the equation. Philip (1955) has reduced the one dimensional form of the equation to an integral form, and has carried out the integration by a forward integration technique and

iteration. He has discussed the solution and its relevant points in great detail. Ashcroft (1962) has written the one dimensional equation in a difference notation and has arrived at a satisfactory solution. Different approaches for the one dimensional case have been furnished by Wang, Hassan, and Franzini (1964) and by Hanks et al (1962) and Rubin et al (1963). At present, a group of researchers[1] at General Electric Company at Hanford, Washington, are working on the numerical solution of the problem of steady unsaturated flows in soils in two dimensions and also in special cases of three dimensions.

2.3 Assumptions in unsaturated flow.

Before considering the solution of the diffusion equation, it is worthwhile to describe the dominant assumptions hidden either in the derivation of the equation or in the solution of it. Some of the assumptions are as follows:

(a) The granular medium through which flow takes place is assumed to be homogeneous. This condition is rarely met in practice. Porosity of soil varies from place to place and the soil particles rarely have a random orientation. However, the situation is not altogether hopeless for in most practical cases, an average or mean representation of the soil may suffice. For this reason, a mean porosity, a median size and a representative shape parameter may be used. Results obtained in this manner generally give reasonably good accuracy. Also, this is

---

[1] Nelson, R. W. (1962) and Reisenauer et al (1963).

the only plausible way to deal with such situations, otherwise the problem will not only be quite involved or tedious, but it may be impossible to handle.

(b) Assumption is also made that the porous medium contains only the flowing liquid under consideration and empty voids, but nothing else. This condition also is not satisfied in practice. Usually air is entrapped in the medium, and it offers resistance to the flow. The resistance to the flow of water through the porous medium by the air depends upon the characteristics of the escape routes available for the air. If the medium is fully ventilated so that air can easily escape when the water front advances, its effect is almost negligible. The air can also escape directly through the advancing liquid if the granular materials are coarse and the surface tension of the liquid is small. In fine soils, when air is confined by an impermeable layer, the compressed air can virtually stop the wetting front from advancing. Adrian (1964) has made such observations in the laboratory on small tubes with closed ends. It is extremely difficult to include the effect of air resistance in the general equation of unsteady and unsaturated flow, and still get a neat solution. Under such circumstances, the assumption of the absence of air in soils is generally made at the present time. Once again the results obtained are not bad even after neglecting the air resistance. The reason is that in most practical situations escape routes are available for the air.

(c) Darcy's law is assumed to be applicable in the derivation of the general equation of unsaturated flow, though the Darcy's law was originally deduced from observations of the flow through saturated

media. It has been found that this assumption is valid within the range of validity of Darcy's law.

(d) As in classical hydrodynamics, the flowing fluids in media are considered to be continuous at microscopic level, when applying continuity in differential form in the derivation of the equation. Also the fluids are considered to be incompressible and the flows to be isothermal. Variation of temperature in porous media might affect diffusivity and hydraulic conductivity; but for small changes in temperature, these assumptions are quite satisfactory.

(e) In the solution of a unsaturated problem, soil properties are taken to be constant. This assumption is certainly far from the truth. As already mentioned, soil has hysterisis characteristics during wetting and drying; that is, values of pF actually change during each cycle of wetting and drying which effects the value of the diffusivity. The assumption that diffusivity is only a function of saturation of the porous media is misleading.

With the application of water, soils swell, and drying of soils decreases their volumes. A thin hygroscopic moisture layer is always present on the surfaces of soil particles, but in certain soils, water may react with the particles to increase or decrease the volume. Removal of moisture from soils decreases their volumes, and the relation between the two is linear at first, but the rate of volume decrease tapers off fast in the end due to entrapped air in soils.

Saturation of soils may aid in aggregation and cementation of soil particles. This changes the hydraulic conductivity and capillary characteristics of the soil. In many soils drying produces wide and

deep cracks because of excessive shrinkage. Moreover, repeated wetting and drying of soils brings about permanent volume increases. A typical curve showing changes in volume due to alternate wetting and drying of soils is given in Fig. 2.4. These effects are not of the same magnitude in all soils. Fine soils like clay are most susceptible to such variations, whereas sand is least affected. Sand has a peculiar property of bulking.

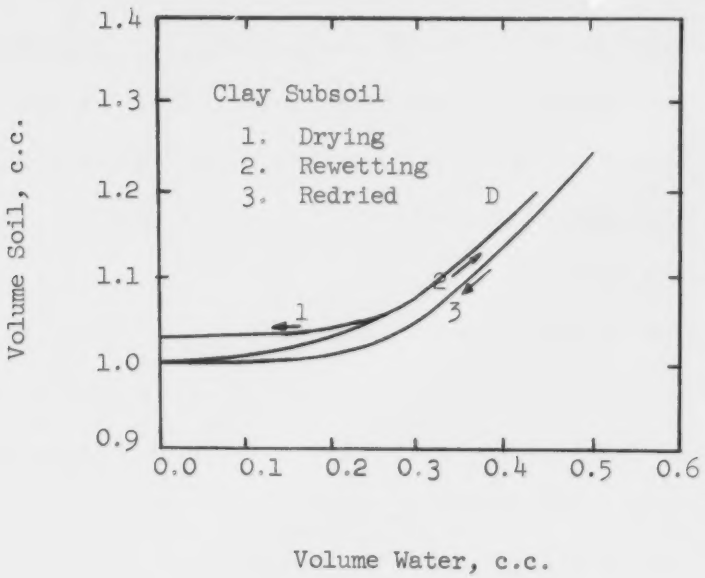


Fig. 2.4: Typical Relationship between Volume Changes and Alternate Wetting and Drying of Soils (After Haines 1925).

Bulking is the increase in volume of the sand when it is partially saturated and stirred. Bulking is different from swelling in the sense that it only takes place when stirred or disturbed whereas swelling can take place in situ. Bulking of sand completely vanishes when it is fully saturated, and in fact it has minimum volume at oversaturation. This property of sand may create a problem in the sampling of partially saturated sands.

The foregoing phenomena can result in a complete change in the pattern of the pore spaces and pore sizes. Some of the pore spaces may become smaller while others are enlarged. Distribution of capillary and non-capillary pores can be radically changed. Consequently, hydraulic conductivity and capillary characteristics of soils undergo continuous change, and in some cases these changes are very substantial. They are difficult to predict and extremely hard to account for in any theoretical consideration.

Apart from changes brought about in soil properties by moisture, complex changes also take place in soils due to vegetation, bacteria and chemicals. Bacteria and microorganisms affect aggregation and porosity of soils. The amount of aggregation of soils is found to be dependent on the concentration of microorganisms among other factors. Vegetation provides channels through the root-zone through which water may percolate. Certain kinds of worms are abundant under thick vegetative cover, and they make the soil highly permeable by channeling it. Changes in soils brought about by biological and bacterial activities are dependent on temperature and the availability of moisture. Seasonal changes in soil properties may be influenced due to this reason.

It is well known that forest and vegetable areas retain a larger quantity of rain water than open and burned land. This also indicates that vegetation and bacterial activity is responsible for higher porosity and permeability of soils. The rate of absorption of water by soils in burned woods has been found to be less than one-fifth of the original rate in some cases.

Depending upon the quantity and character of the water infiltrating through the ground surface, soil pores may get clogged by impurities. Many artificial recharge wells, ditches, and ponds get badly clogged from time to time and they have to be treated at intervals. Old recharge wells have also been abandoned at many places because of their poor performance.

Through chemical action, porosity of soil can become almost zero when particles are cemented together or can become very large as when caves in limestone regions are formed by acidic water.

The presence of an electrolyte in water will alter the permeability of soils. Ruehrwein and Ward (1952) found that polycations are very effective flocculating agents and flock-stabilizing agents for clay. They also found that sodium polymethacrylate functions as a stabilizing agent of clay aggregates by binding the particles together. Similarly, other chemicals present in water and soils may react with each other, and the end result is changes in the soil structure. All these complexities contribute to change in the value of diffusivity for any given soil.

It is obvious that the idealized assumption of constancy of soil properties is not correct, still the results obtained by idealized

solutions are in good agreement with experimental observations in many cases. This is so because the changes in soil properties brought about by the above mentioned causes take place slowly and over long periods of time; they often are not noticeable during a short time interval. In some instances the foregoing factors may tend to compensate for each other and bring about a rough stabilization in soil properties.

CHAPTER 3

HYDRAULIC CONDUCTIVITY AND DIFFUSIVITY OF UNSATURATED SOILS

The diffusion equation contains two terms, the hydraulic conductivity, K and the diffusivity, D which are functions of the degree of saturation of the soil. It is desirable to know the functional relationships between these functions and the saturation of the soil; but exact expressions have not yet been found. However, it would be helpful to study their qualitative properties and some of the empirical relationships that have been suggested.

Hydraulic conductivity of soil is the ratio of the specific discharge to the gradient of the driving force and, as is evident from Darcy's law, it has the dimension of velocity. Many other names also have been used for hydraulic conductivity such as permeability, film conductivity, unsaturated conductivity and capillary conductivity. An effort has been made under the leadership of Richards (1951) to standardize and define terms related to infiltration and drainage, but this recommended terminology has not yet been fully accepted. The writer prefers to use the term "hydraulic conductivity" and will do so in this paper.

Hydraulic conductivity depends on soil properties and temperature of fluid as well as on moisture content. The following comments are applicable to isothermal flow. Flow takes place through a porous medium when the moisture in the pores possesses continuity. Obviously, hydraulic conductivity is highest in a fully saturated soil, when there is a maximum continuity in the moisture. However, all moisture present in the soil does not contribute to the flow, and this portion of the

moisture is called the inactive moisture, θ_i. This moisture is either entrapped in fine pores under the influence of strong molecular forces or is hygroscopic moisture held tightly on the surface of the soil particle. The inactive moisture content is not a constant. Its value decreases with the increase in saturation of soils, because the molecular forces are unbalanced due to thick moisture films.

Another related term is threshold moisture, θ_o defined as that moisture content at which the continuity of moisture is broken and below which flow will not occur. It is a constant for a given soil. When the moisture content θ is equal to the threshold moisture θ_o, then the inactive moisture θ_i is clearly equal to the threshold moisture. A schematic diagram explaining these terms is shown in Fig. 3.1. The value of moisture content at full saturation is denoted by θ_{sat} and, if expressed on a volumetric basis, is equal to the porosity of the medium. Only vapor movement can take place below the threshold moisture content as shown in Fig. 3.1. In this work only the liquid movement will be considered.

Given the dominant factors influencing unsaturated flow, dimensional analysis can be used to obtain a form of functional relationship between hydraulic conductivity and moisture content. Hydraulic conductivity under the above mentioned assumptions depends on unit weight of water γ, the dynamic viscocity μ (since the flow is always very slow), the characteristic diameter of the soil particles d, the porosity p, and the moisture content in the range of liquid movement $\theta-\theta_o$.

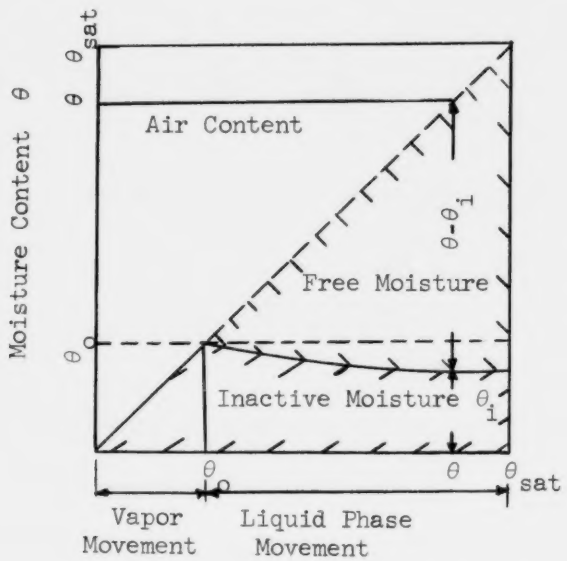


Fig. 3.1: Schematic Representation of Relation between Threshold Moisture Content θ_o and Inactive Moisture Content θ_i (After Wang 1963).

$$K = F(\gamma, \mu, d, p, \theta-\theta_o) \tag{3.1}$$

It can be written as

$$K = C \cdot (\gamma)^a \cdot (\mu)^b \cdot (d)^c \cdot (p)^m \cdot (\theta-\theta_o)^n \tag{3.2}$$

Where C is a constant of proportionality,

and a, b, c, m and n are constant exponents.

It should be noted that the porosity p and $(\theta-\theta_o)$ are dimensionless. Expressing Equation 3.2 in dimensional form gives,

$$\left[\frac{L}{T}\right]^1 = \left[\frac{M}{L^2T^2}\right]^a \cdot \left[\frac{M}{LT}\right]^b \cdot [L]^c \qquad (3.3)$$

solving for a, b, c; Equation (3.2) reduces to

$$K = C \cdot \frac{\gamma}{\mu} d^2 p^m (\theta-\theta_o)^n \qquad (3.4)$$

In Equation (3.4), there are three unknowns C, m and n which can only be determined by experiment. Equation (3.4) can be reduced to a simpler form by substituting θ_{sat} for θ and dividing by the new equation:

$$K = K_{sat}\left(\frac{\theta-\theta_o}{\theta_{sat}-\theta_o}\right)^n \qquad (3.5)$$

In Equation (3.5), there is only one constant n to be determined experimentally; of course K_{sat}, θ_o and θ_{sat} are assumed to be known for a given soil. The writer found that $n = 4.5$ gives a good approximation for Yolo Light Clay for the data given by Moore (1939), as shown in Fig. 3.2. The solid line indicated by 2 in Fig. 3.2 represents the value of K that produces the best fit of the data. Thus,

$$K \approx K_{sat}\left(\frac{\theta-\theta_o}{\theta_{sat}-\theta_o}\right)^{4.5} \qquad (3.6)$$

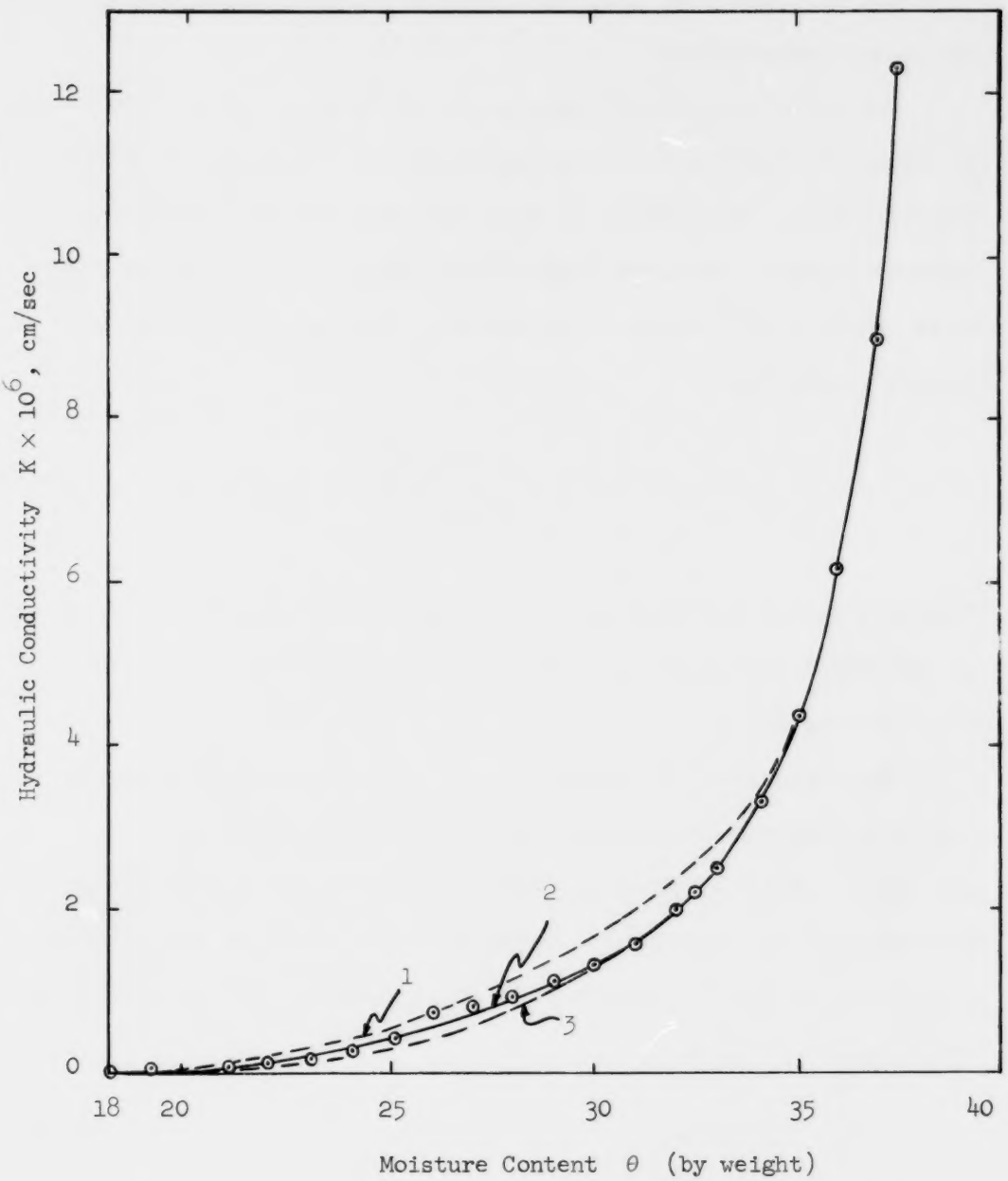


Fig. 3.2: Relation between Hydraulic Conductivity K and Moisture Content θ for Yolo Light Clay (After Moore 1939).

Curves 1 and 3 are plots of Equation 3.5 with the exponents $n = 4$ and $n = 5$ respectively.

For soils the general range of n is between 3 and 6. It is easy to modify the equation for small variations in the values of K at the end points. For example in Yolo Light Clay at the assumed threshold moisture content, the lower limit of hydraulic conductivity was found to be 0.012×10^{-6} cm/sec. The equation (3.6) can be modified and written in the form

$$K = (K_{sat} - 0.012 \times 10^{-6}) \times \left(\frac{\theta - \theta_o}{\theta_{sat} - \theta_o}\right)^{4.5} + 0.012 \times 10^{-6} \qquad (3.7)$$

This is a better approximation satisfying the end conditions. It should be understood that equation (3.5) is only an approximation and may not apply to some soils.

The diffusivity D depends on the hydraulic conductivity, the capillary head and the moisture content through the relation $D = \frac{\partial h_c(\theta)}{\partial \theta} \cdot K(\theta)$ as given in Chapter 2. Since the capillary head and the hydraulic conductivity are functions of moisture content, then the diffusivity is also a function of moisture content. For capillary head, the relation

$$h_c = \frac{A}{\theta} + B \qquad (3.8)$$

where A and B are constants,

has been suggested, but this writer found that this relation is

inadequate unless higher order of terms are taken as follows:

$$h_c = \frac{C_1}{\theta^n} + \cdots\cdots + \frac{C_n}{\theta} + C_{n+1} \qquad (3.9)$$

Equation (3.9) is not very convenient because of the many terms. Also it should be remembered that the relation between h_c and θ depends on whether the soil is drying or wetting.

Many investigators have used exponential relationships for diffusivity. Its general form is

$$D = \alpha e^{\beta\theta} \qquad (3.10)$$

Where α and β are constants to be determined from experimental data.

This writer found the exponential relation of diffusivity to match quite closely the experimental values of diffusivity for Yolo Light Clay as shown in Figure 3.3. This relation has been used in the solution of the diffusion equation which will be explained in detail in later chapters.

E. J. Scott, R. J. Hanks, D. B. Peters, and A. Klute (1962) have solved the one dimensional diffusion equation without gravity by a power series method, using transformations similar to those used by this writer in later chapters. They used exponential relations for diffusivity. They have also used linear relationships for diffusivity, but it does not give a very good fit. They have given two relations for two soils,

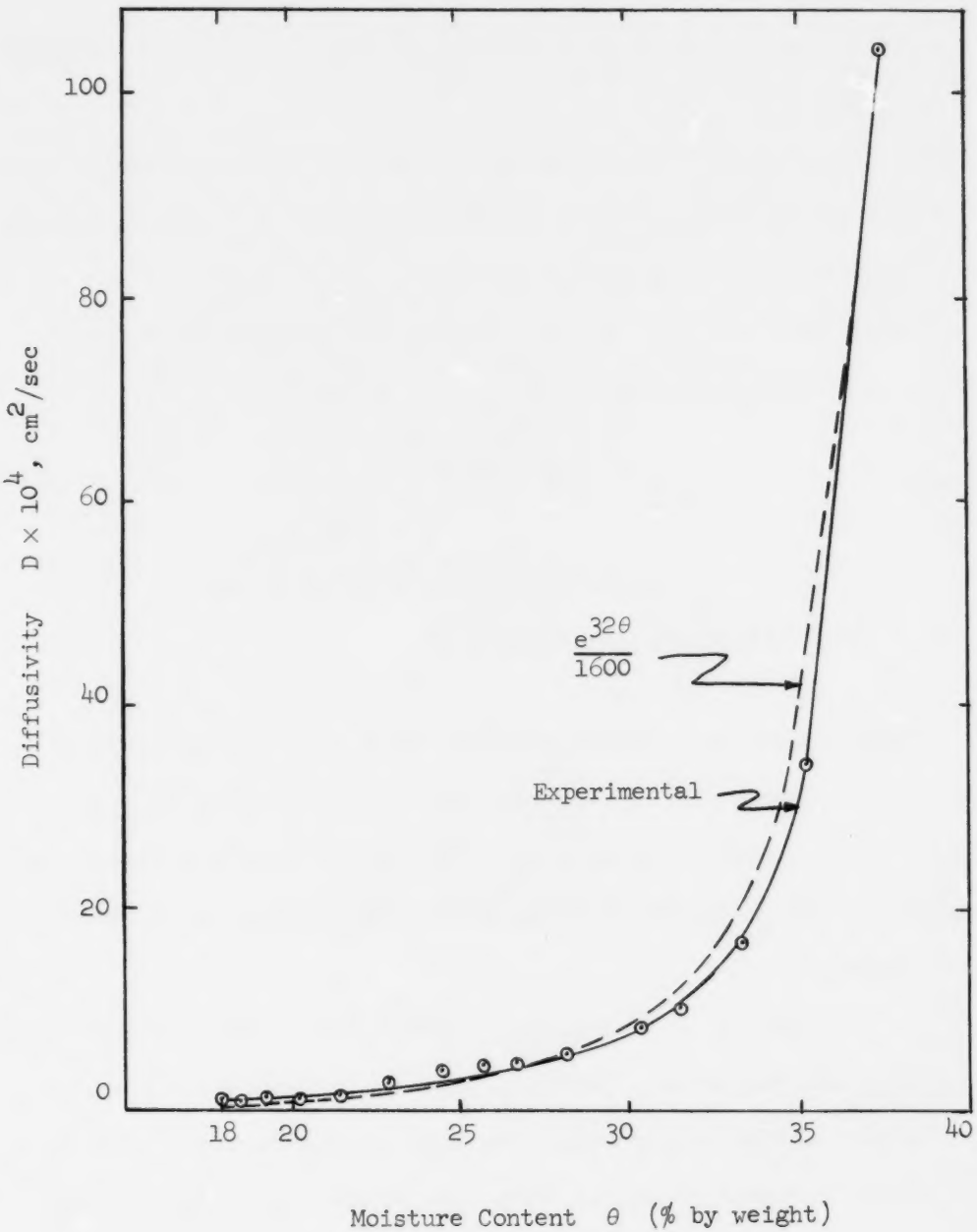


Fig. 3.3: Relation between Diffusivity D and Moisture Content θ for Yolo Light Clay (After Philip 1957).

Sarpy Loam: $D = 1.24 \times 10^{-4} e^{19.78\theta}$ cm^2/sec.

$\theta_{sat} = 0.41$ (Volume basis)

$\theta_o = 0.061$ (Volume basis)

Geary Silt Loam: $D = 5.66 \times 10^{-6} e^{21\theta}$ cm^2/sec.

$\theta_{sat} = 0.386$ (Volume basis)

$\theta_o = 0.053$ (Volume basis)

To further substantiate the exponential relationship for diffusivity, Fig. 3.4 by Ohmstede (1957) is presented. He has plotted diffusivity versus moisture content on semilogarithmic paper for different soils. The experimental points are scattered about straight lines for each soil and clearly show that the exponential relation is a good approximation for them. This writer has calculated the equations of diffusivity for these soils as follows:

Pachappa Soil: $D = 1.25 \times 10^{-3} e^{27.8\theta}$ cm^2/min.

$\theta_{sat} = 0.33$ (Weight basis)

$\theta_o = 0.04$ (Weight basis)

Indio Soil: $D = 1.35 \times 10^{-3} e^{20.4\theta}$ cm^2/min.

$\theta_{sat} = 0.45$ (Weight basis)

$\theta_o = 0.04$ (Weight basis)

Chino Soil: $D = 1.419 \times 10^{-4} e^{18\theta}$ cm^2/min.

$\theta_{sat} = 0.62$ (Weight basis)

$\theta_o = 0.07$ (Weight basis)

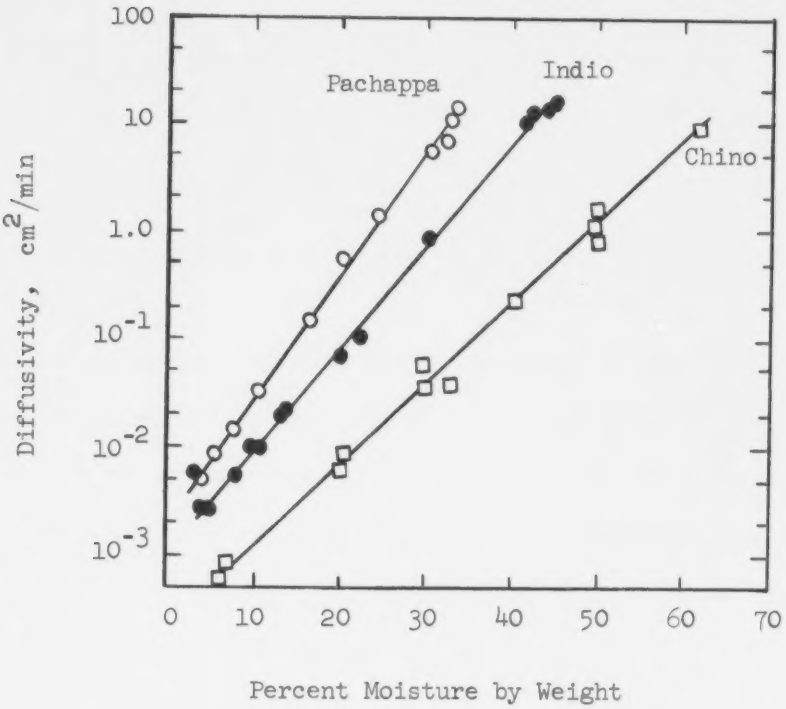


Fig. 3.4: Soil Moisture Diffusivity as a Function of Percent Moisture for Three Soils (After Ohmstede 1957).

If there is difficulty in fitting a curve to diffusivity data with Equation (3.10), then a modified form as follows may be tried.

$$D = \alpha e^{\beta(\theta-\theta_o)} \qquad (3.11)$$

Diffusivity may also be represented by an n-th degree polynomial, but such form is inconvenient.

Since hydraulic conductivity and capillary head are both functions of moisture content, an effort has been made to relate them. Ohmstede (1957) has plotted curves relating these two functions. A plot of hydraulic conductivity versus capillary head for Yolo Light Clay is shown in Figure 3.5.

No satisfactory analytic relationship between capillary head and diffusivity is known. An approximate relationship however can be found by using the third degree relationship of Equation (3.9) i.e.,

$$h_c = \frac{A}{\theta^3} + \frac{B}{\theta^2} + \frac{C}{\theta} + E \qquad (3.12)$$

From Equation 3.10

$$\theta = \frac{1}{\beta} \ln \frac{D}{\alpha} \qquad (3.13)$$

Substituting (3.13) into (3.12),

$$h_c = \frac{A_1}{(\ln \frac{D}{\alpha})^3} + \frac{B_1}{(\ln \frac{D}{\alpha})^2} + \frac{C_1}{\ln \frac{D}{\alpha}} + E \qquad (3.14)$$

where $A_1 = A\beta^3$

$B_1 = B\beta^2$ and $C_1 = C\beta$.

Equation (3.14) gives a satisfactory fit for Yolo Light Clay, and it may be useful for other soils also.

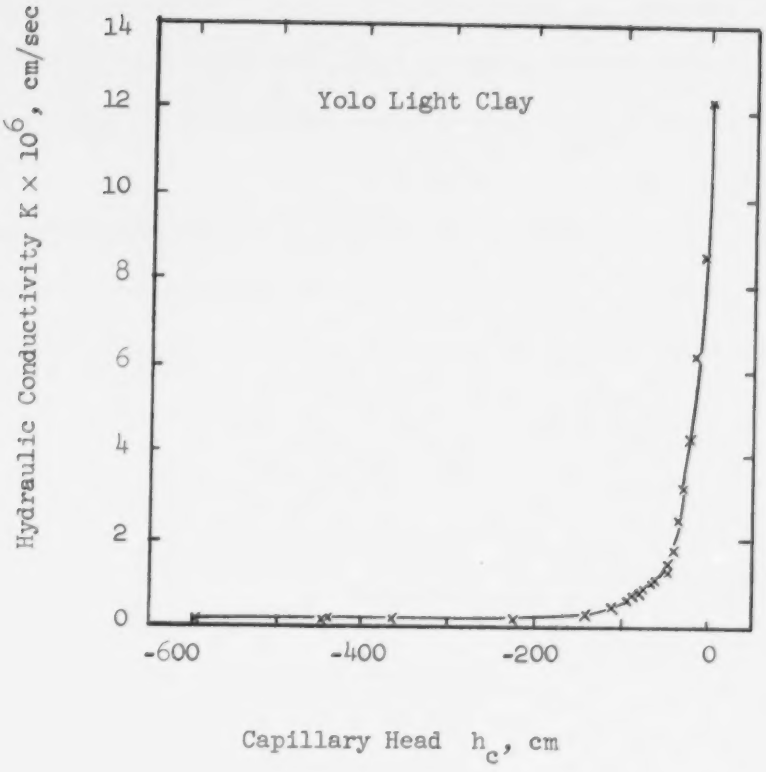


Fig. 3.5: Relationship between Hydraulic Conductivity and Capillary Head (After Philip (1957).

CHAPTER 4

TRANSFORMATION OF DIFFUSION EQUATION INTO POLAR COORDINATES

The diffusion equation may be used to solve the two dimensional problem of outward movement of water from a finite cylindrical source. In this case, it is convenient to use the diffusion equation in polar coordinates. Rewriting the two dimensional form of Equation (2.7),

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta)\,\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(D(\theta)\,\frac{\partial \theta}{\partial y}\right) - \frac{\partial K(\theta)}{\partial y} \qquad (4.1)$$

Here y is taken as positive upwards in the vertical direction. Expanding equation (4.1)

$$\frac{\partial \theta}{\partial t} = \frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + D\,\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} + D\,\frac{\partial^2 \theta}{\partial y^2} - \frac{\partial K}{\partial y}$$

or
$$-\frac{\partial K}{\partial y} + D\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \left(\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y}\right) = \frac{\partial \theta}{\partial t} \qquad (4.2)$$

In Equation (4.2), $K(\theta)$ and $D(\theta)$ have been written simply as K and D, but these functions should always be understood to be dependent on θ.

To transform Equation (4.2) into polar coordinates, the following partial derivatives are written (see Figure 4.1).

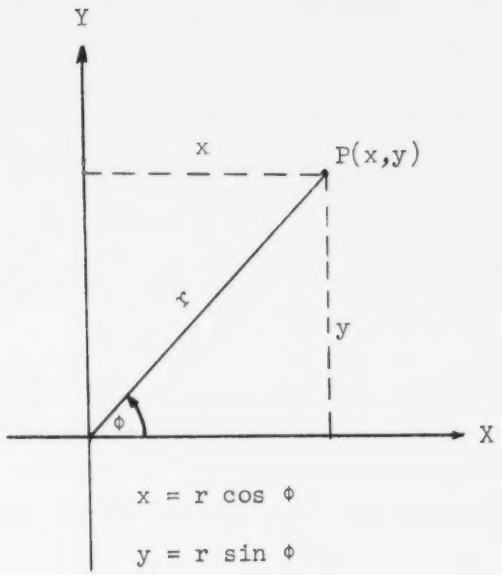


Fig. 4.1: Relation between Cartesian and Polar Coordinates.

$$r = f(x,y)$$

$$\therefore \quad \frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \theta}{\partial y} \cdot \frac{\partial y}{\partial r}$$

but $$\frac{\partial x}{\partial r} = \frac{\partial (r \cos \phi)}{\partial r} = \cos \phi$$

and $$\frac{\partial y}{\partial r} = \frac{\partial (r \sin \phi)}{\partial r} = \sin \phi$$

$$\therefore \quad \frac{\partial \theta}{\partial r} = \cos \phi \frac{\partial \theta}{\partial x} + \sin \phi \frac{\partial \theta}{\partial y} \qquad (4.3)$$

Similarly because $$\phi = g(x,y)$$

$$\therefore \quad \frac{\partial \theta}{\partial \phi} = \frac{\partial \theta}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial \theta}{\partial y} \cdot \frac{\partial y}{\partial \phi}$$

but $$\frac{\partial x}{\partial \phi} = -r \sin \phi$$

and $$\frac{\partial y}{\partial \phi} = r \cos \phi$$

$$\therefore \quad \frac{\partial \theta}{\partial \phi} = -r \sin \phi \frac{\partial \theta}{\partial x} + r \cos \phi \frac{\partial \theta}{\partial y} \tag{4.4}$$

Solving equation (4.3)

$$\frac{\partial \theta}{\partial y} = \frac{1}{\sin \phi} \{\frac{\partial \theta}{\partial r} - \cos \phi \frac{\partial \theta}{\partial x}\} \tag{4.5}$$

Substituting in equation (4.4)

$$\frac{\partial \theta}{\partial \phi} = -r \sin \phi \cdot \frac{\partial \theta}{\partial x} + \frac{r \cos \phi}{\sin \phi} \{\frac{\partial \theta}{\partial r} - \cos \phi \frac{\partial \theta}{\partial x}\}$$

$$= -r \sin \phi \cdot \frac{\partial \theta}{\partial x} + \frac{r \cos \phi}{\sin \phi} \cdot \frac{\partial \theta}{\partial r} - \frac{r \cos^2 \phi}{\sin \phi} \frac{\partial \theta}{\partial x}$$

using the relation $\cos^2 \phi = 1 - \sin^2 \phi$,

$$r \sin \phi \cdot \frac{\partial \theta}{\partial x} + \frac{r}{\sin \phi} \cdot \frac{\partial \theta}{\partial x} - r \sin \phi \frac{\partial \theta}{\partial x} = \frac{r \cos \phi}{\sin \phi} \cdot \frac{\partial \theta}{\partial r} - \frac{\partial \theta}{\partial \phi}$$

Multiplying both sides by $\frac{\sin \phi}{r}$

$$\frac{\partial \theta}{\partial x} = \cos \phi \frac{\partial \theta}{\partial r} - \frac{\sin \phi}{r} \frac{\partial \theta}{\partial \phi} \tag{4.6}$$

Substituting Equation (4.6) into Equation (4.5) gives

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sin\phi}\left\{\frac{\partial\theta}{\partial r} - \cos\phi\left(\cos\phi \cdot \frac{\partial\theta}{\partial r} - \frac{\sin\phi}{r} \cdot \frac{\partial\theta}{\partial\phi}\right)\right\}$$

or

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sin\phi} \cdot \frac{\partial\theta}{\partial r} - \frac{\cos^2\phi}{\sin\phi} \cdot \frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial\theta}{\partial\phi}\cos\phi$$

$$= \frac{1}{\sin\phi} \cdot \frac{\partial\theta}{\partial r} - \frac{1}{\sin\phi} \cdot \frac{\partial\theta}{\partial r} + \sin\phi\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial\theta}{\partial\phi}\cos\phi$$

Finally,

$$\therefore \quad \frac{\partial\theta}{\partial y} = \sin\phi\frac{\partial\theta}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial\theta}{\partial\phi} \qquad (4.7)$$

To obtain $-\frac{\partial K}{\partial y}$, θ in Equation (4.7) is replaced by K and the sign is changed, then

$$-\frac{\partial K}{\partial y} = -\left[\sin\phi\frac{\partial K}{\partial r} + \frac{\cos\phi}{r}\frac{\partial K}{\partial\phi}\right] \qquad (4.8)$$

Similarly $\frac{\partial D}{\partial x}$ and $\frac{\partial D}{\partial y}$ can be found by replacing θ by D in Equations (4.6) and (4.7)

$$\therefore \quad \left[\frac{\partial D}{\partial x} \cdot \frac{\partial\theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial\theta}{\partial y}\right] = \left[\left\{\cos\phi \cdot \frac{\partial D}{\partial r} - \frac{\sin\phi}{r}\frac{\partial D}{\partial\phi}\right\} \cdot \left\{\cos\phi\frac{\partial\theta}{\partial r} - \frac{\sin\phi}{r} \cdot \frac{\partial\theta}{\partial\phi}\right\} + \left\{\sin\phi\frac{\partial D}{\partial r} + \frac{\cos\phi}{r} \cdot \frac{\partial D}{\partial\phi}\right\} \cdot \left\{\sin\phi \cdot \frac{\partial\theta}{\partial r} + \frac{\cos\phi}{r} \cdot \frac{\partial\theta}{\partial\phi}\right\}\right]$$

$$= \cos^2\phi \cdot \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial r} - \frac{\sin\phi\cos\phi}{r} \cdot \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial \phi} - \frac{\sin\phi\cos\phi}{r} \cdot \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial r}$$

$$+ \frac{\sin^2\phi}{r^2} \cdot \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial \phi} + \sin^2\phi \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial r} + \frac{\sin\phi\cos\phi}{r} \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial \phi}$$

$$+ \frac{\sin\phi\cos\phi}{r} \cdot \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial r} + \frac{\cos^2\phi}{r^2} \cdot \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial \phi}$$

$$\therefore \left[\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y}\right] = \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial r} \{\sin^2\phi + \cos^2\phi\} + \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial \phi} \left(\frac{\sin^2\phi + \cos^2\phi}{r^2}\right)$$

or
$$\left[\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y}\right] = \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial r} + \frac{1}{r^2} \frac{\partial D}{\partial \phi} \cdot \frac{\partial \theta}{\partial \phi} \qquad (4.9)$$

$\frac{\partial \theta}{\partial t}$ on the right side of Equation (4.2) does not need transformation, because this is a differentiation with respect to time, not involving x and y in cartesian coordinates nor r and ϕ in polar coordinates.

To find $\frac{\partial^2 \theta}{\partial x^2}$, $\frac{\partial \theta}{\partial x}$ can be substituted into Equation (4.6) in place of θ,

$$\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \cos\phi \frac{\partial}{\partial r}\left(\frac{\partial \theta}{\partial x}\right) - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi}\left(\frac{\partial \theta}{\partial x}\right)$$

Now substituting Equation (4.6) for $\frac{\partial \theta}{\partial x}$ and differentiating

$$\frac{\partial^2 \theta}{\partial x^2} = \cos\phi \frac{\partial}{\partial r}\left(\cos\phi \frac{\partial \theta}{\partial r} - \frac{\sin\phi}{r} \frac{\partial \theta}{\partial \phi}\right) - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial \theta}{\partial r} - \frac{\sin\phi}{r} \cdot \frac{\partial \theta}{\partial \phi}\right)$$

$$\text{or} \quad \frac{\partial^2\theta}{\partial x^2} = \cos\Phi \left\{\cos\Phi \frac{\partial^2\theta}{\partial r^2}\right) - \sin\Phi \cos\Phi \frac{\partial(\frac{1}{r})}{\partial r} \cdot \frac{\partial\theta}{\partial\Phi} - \frac{\cos\Phi \sin\Phi}{r} \frac{\partial}{\partial r}\left(\frac{\partial\theta}{\partial\Phi}\right)$$

$$- \frac{\sin\Phi}{r} \cos\Phi \frac{\partial}{\partial\Phi}\left(\frac{\partial\theta}{\partial r}\right) - \frac{\sin\Phi}{r} \cdot \frac{\partial \cos\Phi}{\partial\Phi} \cdot \frac{\partial\theta}{\partial r}$$

$$+ \frac{\sin\Phi}{r^2} \frac{\partial \sin\Phi}{\partial\Phi} \cdot \left(\frac{\partial\theta}{\partial\Phi}\right) + \frac{\sin^2\Phi}{r^2} \frac{\partial}{\partial\Phi}\left(\frac{\partial\theta}{\partial\Phi}\right)$$

$$\text{or} \quad \frac{\partial^2\theta}{\partial x^2} = \cos^2\Phi \frac{\partial^2\theta}{\partial r^2} + \frac{\sin\Phi \cos\Phi}{r^2} \frac{\partial\theta}{\partial\Phi} - \frac{\sin\Phi \cos\Phi}{r} \cdot \frac{\partial^2\theta}{\partial\Phi \cdot \partial r}$$

$$- \frac{\sin\Phi \cos\Phi}{r} \frac{\partial^2\theta}{\partial r \cdot \partial\Phi} + \frac{\sin^2\Phi}{r} \cdot \frac{\partial\theta}{\partial r} + \frac{\sin^2\Phi}{r^2} \frac{\partial^2\theta}{\partial\Phi^2} + \frac{\sin\Phi \cos\Phi}{r^2} \cdot \frac{\partial\theta}{\partial\Phi} \tag{4.10}$$

Similarly by using Equation (4.7)

$$\frac{\partial^2\theta}{\partial y^2} = \sin^2\Phi \frac{\partial^2\theta}{\partial r^2} - \frac{1}{r^2} \cos\Phi \sin\Phi \frac{\partial\theta}{\partial\Phi} + \frac{\sin\Phi \cos\Phi}{r} \frac{\partial^2\theta}{\partial\Phi \cdot \partial r}$$

$$+ \frac{1}{r} \cos^2\Phi \frac{\partial\theta}{\partial r} + \frac{\sin\Phi \cos\Phi}{r} \frac{\partial^2\theta}{\partial r \cdot \partial\Phi} - \frac{\sin\Phi \cos\Phi}{r^2} \frac{\partial\theta}{\partial\Phi}$$

$$+ \frac{\cos^2\Phi}{r^2} \frac{\partial^2\theta}{\partial\Phi^2} \quad . \tag{4.11}$$

Adding Equations (4.10) and (4.11) and cancelling similar terms with opposite signs, gives

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\theta}{\partial r^2}(\sin^2\Phi + \cos^2\Phi) + \frac{1}{r} \cdot \frac{\partial\theta}{\partial r}(\sin^2\Phi + \cos^2\Phi)$$

$$+ \frac{1}{r^2} \frac{\partial^2\theta}{\partial^2\Phi}(\sin^2\Phi + \cos^2\Phi)$$

or
$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2\theta}{\partial\phi^2} \tag{4.12}$$

Adding Equations (4.8), (4.12) multiplied by D and (4.9),

$$- \frac{\partial K}{\partial y} + D\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \left(\frac{\partial D}{\partial x} \cdot \frac{\partial\theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial\theta}{\partial y}\right) =$$

$$- \left[\sin\phi \frac{\partial K}{\partial r} + \frac{\cos\phi}{r} \frac{\partial K}{\partial\phi}\right] + D\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r} \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2\theta}{\partial\phi^2}\right]$$

$$+ \left[\frac{\partial D}{\partial r} \cdot \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \frac{\partial D}{\partial\phi} \cdot \frac{\partial\theta}{\partial\phi}\right] \tag{4.13}$$

Comparing Equations (4.2) and (4.13), the general two dimensional diffusion equation in polar coordinates becomes,

$$- \left[\sin\phi \frac{\partial K}{\partial r} + \frac{\cos\phi}{r} \frac{\partial K}{\partial\phi}\right] + D\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r} \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2\theta}{\partial\phi^2}\right]$$

$$+ \left[\frac{\partial D}{\partial r} \cdot \frac{\partial\theta}{\partial r} + \frac{1}{r^2} \frac{\partial D}{\partial\phi} \cdot \frac{\partial\theta}{\partial\phi}\right] = \frac{\partial\theta}{\partial t} \tag{4.14}$$

Equation (4.14) can be written in an alternate form

$$\frac{D}{r} \frac{\partial\theta}{\partial r} + \frac{\partial}{\partial r}\left(D \cdot \frac{\partial\theta}{\partial r}\right) + \frac{\partial}{r^2\partial\phi}\left(D \frac{\partial\theta}{\partial\phi}\right)$$

$$-\left(\sin\phi \frac{\partial K}{\partial r} + \frac{1}{r} \cos\phi \frac{\partial K}{\partial\phi}\right) = \frac{\partial\theta}{\partial t} \tag{4.15}$$

Equation (4.15) can be further reduced to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(Dr\frac{\partial\theta}{\partial r}\right)+\frac{\partial}{r^2\partial\phi}\left(D\frac{\partial\theta}{\partial\phi}\right)-\left(\sin\phi\frac{\partial K}{\partial r}+\frac{1}{r}\cos\phi\frac{\partial K}{\partial\phi}\right)=\frac{\partial\theta}{\partial t} \quad (4.16)$$

This equation is the final form of the diffusion equation in polar coordinates; it will be referred to in Chapter 5 and 6.

CHAPTER 5

SOLUTION OF THE DIFFERENCE EQUATION BY IMPLICIT FINITE DIFFERENCE METHOD

The solution of the general two dimensional diffusion equation was first attempted by an implicit finite difference method. For this, Equation (4.15) was written into a difference equation.

In Figure 5.1, r, Φ and θ denote radius, angle in counter-clockwise direction from x axis and the saturation of the soil respectively; Δr, $\Delta\Phi$ and Δt are used for their increments. A grid in polar coordinates is made with equal Δr and $\Delta\Phi$. Each intersection of the grid is denoted by two numbers k and m, indicating a particular interval of r and Φ respectively, and they are used as subscripts. A superscript n is used to show a particular interval of time with θ, K and D. The radius of finite cylindrical source is taken as r_o.

Using the relations

$$\left.\frac{\partial\theta}{\partial r}\right|_{k,m}^{n} = \frac{\theta^{n}_{k+1,m} - \theta^{n}_{k-1,m}}{2 \cdot \Delta r} \quad ,$$

$$\left.\frac{\partial\theta}{\partial\Phi}\right|_{k,m}^{n} = \frac{\theta^{n}_{k,m+1} - \theta^{n}_{k,m-1}}{2 \cdot \Delta\Phi} \quad ,$$

$$\left.\frac{\partial^2\theta}{\partial r^2}\right|_{k,m}^{n} = \frac{\dfrac{\theta^{n}_{k+1,m}-\theta^{n}_{k,m}}{\Delta r} - \dfrac{\theta^{n}_{k,m}-\theta^{n}_{k-1,m}}{\Delta r}}{\Delta r}$$

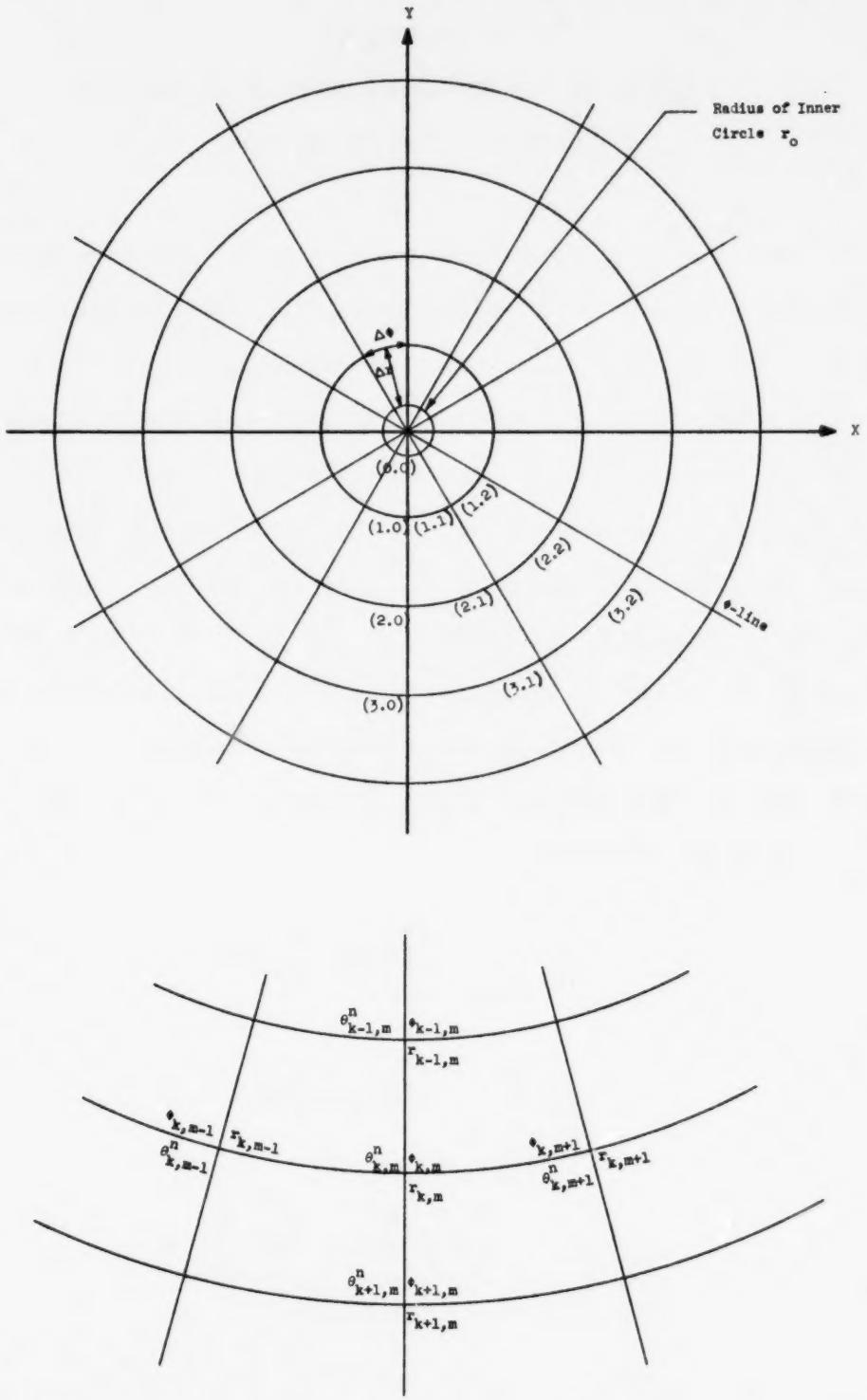


Fig. 5.1: Grid for Difference Equation

and

$$\left.\frac{\partial^2\theta}{\partial\phi^2}\right|^{n}_{k,m} = \frac{\frac{\theta^{n}_{k,m+1}-\theta^{n}_{k,m}}{\Delta\Phi} - \frac{\theta^{n}_{k,m}-\theta^{n}_{k,m-1}}{\Delta\Phi}}{\Delta\Phi} \quad ;$$

Equation (4.15) can be written

$$\frac{D^{n+\frac{1}{2}}_{k,m}}{r_{k,m}} \cdot \frac{\theta^{n+1}_{k+1,m}-\theta^{n+1}_{k-1,m}}{2\cdot\Delta r} + \frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}\,\frac{\theta^{n+1}_{k+1,m}-\theta^{n+1}_{k,m}}{\Delta r} - D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}\,\frac{\theta^{n+1}_{k,m}-\theta^{n+1}_{k-1,m}}{\Delta r}}{\Delta r}$$

$$+ \frac{1}{r^2_{k,m}} \cdot \frac{D^{n+\frac{1}{2}}_{k,m+\frac{1}{2}}\,\frac{\theta^{n+1}_{k,m+1}-\theta^{n+1}_{k,m}}{\Delta\Phi} - D^{n+\frac{1}{2}}_{k,m-\frac{1}{2}}\,\frac{\theta^{n+1}_{k,m}-\theta^{n+1}_{k,m-1}}{\Delta\Phi}}{\Delta\Phi}$$

$$- \left[\sin\Phi_{k,m}\,\frac{K^{n+\frac{1}{2}}_{k+1,m}-K^{n+\frac{1}{2}}_{k-1,m}}{2\cdot\Delta r} + \frac{1}{r_{k,m}}\cos\Phi_{k,m}\,\frac{K^{n+\frac{1}{2}}_{k,m+1}-K^{n+\frac{1}{2}}_{k,m-1}}{2\cdot\Delta\Phi}\right]$$

$$= \frac{\theta^{n+1}_{k,m}-\theta^{n}_{k,m}}{\Delta t} \qquad (5.1)$$

Solving and writing Equation (5.1) in a useful form,

$$\theta^{n+1}_{k,m}\left[-\frac{1}{\Delta t} - \frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}}{(\Delta r)^2} - \frac{D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2} - \frac{D^{n+\frac{1}{2}}_{k,m+\frac{1}{2}}}{r^2_{k,m}(\Delta\Phi)^2} - \frac{D^{n+\frac{1}{2}}_{k,m-\frac{1}{2}}}{r^2_{k,m}(\Delta\Phi)^2}\right]$$

$$+ \theta^{n+1}_{k-1,m}\left[-\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r} + \frac{D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2}\right] + \theta^{n+1}_{k+1,m}\left[\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r} + \frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}}{(\Delta r)^2}\right]$$

$$+ \theta^{n+1}_{k,m-1} \left[\frac{D^{n+\frac{1}{2}}_{k,m-\frac{1}{2}}}{r^2_{k,m}(\Delta\Phi)^2} \right] + \theta^{n+1}_{k,m+1} \left[\frac{D^{n+\frac{1}{2}}_{k,m+\frac{1}{2}}}{r^2_{k,m}(\Delta\Phi)^2} \right]$$

$$+ \left[\frac{\theta^{n}_{k,m}}{\Delta t} - \sin \Phi_{k,m} \frac{K^{n+\frac{1}{2}}_{k-1,m} - K^{n+\frac{1}{2}}_{k+1,m}}{2 \cdot \Delta r} - \cos \Phi_{k,m} \frac{K^{n+\frac{1}{2}}_{k,m-1} - K^{n+\frac{1}{2}}_{k,m+1}}{2r_{k,m} \cdot \Delta\Phi} \right] = 0$$

(5.2)

To solve Equation (5.2), a polar grid of desired radius and polar angle interval was put on the region, and the finite difference equation was written for each grid intersection, obtaining a system of linear equations of a number equal to the total numbers of grid intersections at any time interval. Only half of the region was considered for the solution of the problem, because the flow is symmetric with respect to the vertical axis.

There are two difficulties in starting this procedure:

1. It is not known how far the wetting front will move from the initial radius of the cylindrical source r_o to a new radius r_1 in the first time interval. The part of the region selected for iteration for each time interval must cover at least up to the maximum position of the wetting front at the end of that time interval. The writer simply guessed the distance to the wetting front at the end of each time interval. This does not present any difficulty, because the assumed r_1 can be equal to or greater than the actual r_1 due to the constancy of the solution beyond the actual location of the wetting front, as θ does not change beyond it. Care must be taken not to underestimate the values

of r_1, otherwise the solution will be in error. For this, a few trials may be needed until the asymptotic behavior of the solution is noticed.

2. The two terms $D(\theta)$ and $K(\theta)$ which are present in the difference equation depend on the values of θ which are obtained from the solution. The values of θ for each grid point at the beginning of each time interval are assumed. With the assumed values of θ, the corresponding values of D and K as obtained from the experimental data were used for each grid point of the region and then the system of simultaneous equations were solved by the Gauss Elimination method. The new values of θ thus obtained for each grid intersection were compared with the assumed values of θ. If the differences between the two sets of θ values were not within the tolerable limit, then the procedure was repeated with the new values of θ. This iterative procedure has to be repeated for each time interval.

The writer abandoned this method after considerable work for the following reasons:

a. The method does not seem to have a good convergent characteristic unless Δr, $\Delta\phi$ and Δt are very small. In case of a rough grid (having comparatively large values of Δr and $\Delta\phi$), the convergence of the equation grows worse with increasing time.

b. The method is very lengthy and time-consuming even with fast modern electronic computers. For example, the writer applied the above method to a rough grid of $\Delta r = 2$ cm, $\Delta\phi = 30^\circ$ and $\Delta t = 3.75 \times 10^4$ seconds. For the first time interval, 35 simultaneous equations on the desired region were obtained. These equations had to be solved at

least 15 times successively during the iteration to get a satisfactory solution for the first time interval. As the number of time intervals increased, the region expanded and the number of simultaneous equations also increased. Even with this rough grid, the Burroughs B 5500 digital computer could not complete the computations for three time intervals in less than five minutes. The writer estimated that the time required by the computer for each time interval grows approximately exponentially with the number of time intervals.

Obviously, a finer mesh for convergence of the solution at higher time intervals will require even larger times on the computer and thus this method is impracticable. The writer suspects that appreciable rounding errors are accumulated in this method by the very large number of computations involved at each step. This method might be improved by further investigation.

Due to the impracticality of the difference method, another method was devised and used to solve this problem. This method, which is detailed in later chapters, is very satisfactory.

CHAPTER 6

TRANSFORMATION OF THE DIFFUSION EQUATION TO AN ORDINARY DIFFERENTIAL EQUATION

The Diffusion Equation (4.14) is extremely difficult to solve, and some manipulation is necessary to get a meaningful solution. Re-writing Equation (4.14)

$$-\left[\sin\Phi\,\frac{\partial K}{\partial r}+\frac{\cos\Phi}{r}\,\frac{\partial K}{\partial\Phi}\right]+D\left[\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\,\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2\theta}{\partial\Phi^2}\right]$$

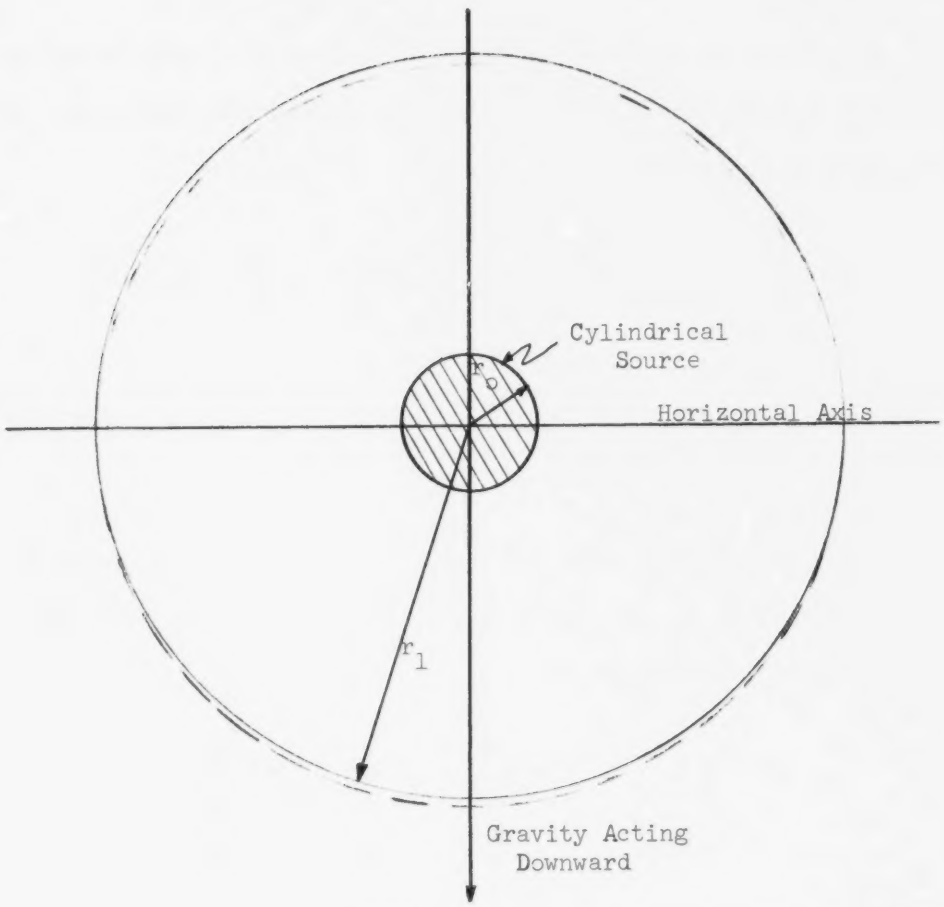
$$+\left[\frac{\partial D}{\partial r}\cdot\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\,\frac{\partial D}{\partial\Phi}\cdot\frac{\partial\theta}{\partial\Phi}\right]=\frac{\partial\theta}{\partial t} \qquad (6.1)$$

This equation describes the flow of liquid outward through an unsaturated region from a finite cylindrical source. In this case, there are two boundary conditions,

$$\left.\begin{aligned} r = r_o\ ,\quad \theta = \theta_{sat} \quad &\text{for} \quad t > 0 \\ r = \infty\ ,\quad \theta = \theta_n \quad &\text{for} \quad t > 0 \end{aligned}\right\} \qquad (6.2)$$

where r_o is the radius of the finite cylindrical source and θ_n is the initial moisture content in the soil.

The solution of the general equation including the effect of gravity would represent a flow whose wetting front would be symmetric with respect to the vertical axis passing through the center of the source, but would be unsymmetric with respect to any other axis as shown in Figure 6.1.



Solid Line: Wetting Front
without Gravity

Broken Line: Wetting Front
with Gravity

Fig. 6.1: Effect of Gravity.

If the influence of gravity were neglected, the flow would be completely symmetric with respect to the axis passing through the center of the source, and perpendicular to the paper. The wetting front thus would make an exact circle.

Actually the effect of gravity is usually small compared to that of capillarity, and no great error results if an approximation is made neglecting gravity. This is particularly true if the soil is made up of fine particles, i.e., a silt or clay. Many researchers in the past have approximated the solution of the general one dimensional diffusion equation by neglecting gravity.

In the analysis here the effect of gravity will be neglected initially to achieve a simplification in the solution of the diffusion equation. An approximate correction for the effect of gravity will be applied later.

After neglecting gravity, the solution becomes independent of Φ due to symmetry, and Equations (6.1) and (6.2) reduce to

$$D \frac{\partial^2 \theta}{\partial r^2} + \frac{D}{r} \frac{\partial \theta}{\partial r} + \frac{\partial D}{\partial r} \cdot \frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial t} \tag{6.3a}$$

with boundary conditions

$$\left.\begin{aligned} r = r_o \ , \quad \theta = \theta_{sat} \quad &\text{for} \quad t > 0 \\ r = \infty \ , \quad \theta = \theta_n \quad &\text{for} \quad t > 0 \end{aligned}\right\} \tag{6.3b}$$

Making a change in variables

$$\left.\begin{aligned} \xi &= \frac{r}{\sqrt{2t}} \\ \text{and} \qquad \psi(\xi) &= \theta(r,t) \end{aligned}\right\} \tag{6.4}$$

the following derivatives are obtained,

$$\frac{\partial\theta}{\partial r} = \frac{\partial\theta}{\partial\xi} \cdot \frac{\partial\xi}{\partial r} = \frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}}$$

$$\frac{\partial^2\theta}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial\xi}\left(\frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}}\right) \cdot \frac{\partial\xi}{\partial r}$$

$$= \frac{1}{2t} \frac{d^2\psi}{\partial\xi^2}$$

$$\frac{\partial D}{\partial r} = \frac{\partial D}{\partial\theta} \cdot \frac{\partial\theta}{\partial r} = \frac{dD}{d\theta} \cdot \frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}}$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial\xi} \cdot \frac{\partial\xi}{\partial t} = \frac{d\psi}{d\xi} \cdot \left(-\frac{r}{\sqrt{2t^3}}\right) = \frac{d\psi}{d\xi} \cdot \left(-\frac{\xi\sqrt{2t}}{\sqrt{2t^3}}\right) = -\frac{\xi}{2t}\frac{d\psi}{d\xi}$$

Substituting the above derivatives into Equation (6.3),

$$D \cdot \frac{1}{2t} \cdot \frac{d^2\psi}{d\xi^2} + \frac{D}{\xi\cdot\sqrt{2t}} \cdot \frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}} + \frac{dD}{d\theta} \cdot \frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}} \cdot \frac{d\psi}{d\xi} \cdot \frac{1}{\sqrt{2t}}$$

$$= -\frac{\xi}{2t} \cdot \frac{d\psi}{d\xi}$$

With the abbreviations

$$D' = \frac{dD}{d\theta}, \quad \psi' = \frac{d\psi}{d\xi} \quad \text{and} \quad \psi'' = \frac{d^2\psi}{d\xi^2}$$

the equation can be written as

$$D\psi'' + \frac{D}{\xi}\psi' + D'\psi'^2 + \xi\psi' = 0 \qquad (6.5.a)$$

Similarly boundary conditions take the form for $t > 0$,

$$\left.\begin{aligned} \xi = \xi_o = \frac{r_o}{\sqrt{2t_o}}, \quad & \psi(\xi_o) = \theta_{sat} \\ \xi = \infty, \quad & \psi(\infty) = \theta_n \end{aligned}\right\} \qquad (6.5.b)$$

Subscript "o" refers to the initial values of the quantities subscripted.

Equation (6.5.a) has a special property which should be pointed out. This equation is exact for flow without gravity and no condition has been imposed on D. It may be possible to integrate the equation numerically using the actual values of D as obtained from experiment. The method by Runge-Kutta (1901) may be a good technique in this case. This writer did not choose this method for two reasons:

1. In Equation (6.5.a), the values of $\frac{dD}{d\theta}$ are needed. If a curve of D versus θ is plotted and then slope is measured from the curve, inaccuracies will result as it is extremely difficult to obtain a satisfactory solution from derivatives measured from curves.

2. The Runge-Kutta method is based on a 4^{th} order Taylor's series and for a large number of steps, errors that accumulate due to rounding and the neglected 5^{th} order term become appreciable. There is always the possibility of improving the solution by taking very small steps, and this is a good area for future research.

Equation (6.5) can be written in a different form. Noting that

$$\frac{d}{d\xi}(D\psi') = D\psi'' + D'\psi'^2$$

the equation reduces to

$$\frac{d}{d\xi}(D\psi') + \frac{D}{\xi}\psi' + \xi\psi' = 0$$

or
$$\xi\frac{d}{d\xi}(D\psi') + D\psi' + \xi^2\psi' = 0$$

or
$$\frac{d}{d\xi}(\xi D\psi') + \xi^2\psi' = 0 \qquad (6.6)$$

Finally Equation (6.6) can be written in an integral form

$$\xi D\psi' = -\int_{\xi_o}^{\xi} \xi^2\psi' \qquad (6.7)$$

Equation (6.7) is similar to an integral equation obtained by Philip (1957) for the one dimensional case, and his technique of approximation by a series can be used in this case, but the writer chose not to use this because of the limited radius of convergence of the series as explained by Philip. Beyond the radius of convergence of the series, some kind of assumption and approximation has to be made. This may be another good area for future research.

To proceed ahead with Equation (6.5), the writer has used a relation for D as explained in Chapter 3. Taking $D = \alpha e^{\beta\theta}$ and substituting in Equation (6.5)

$$\alpha e^{\beta\theta}{}_{\psi}'' + \frac{\alpha e^{\beta\theta}}{\xi}\,\psi' + \alpha\beta e^{\beta\theta}{}_{\psi}'^{2} + \xi\psi' = 0$$

or
$$\alpha e^{\beta\psi}{}_{\psi}'' + \frac{\alpha e^{\beta\psi}}{\xi}\,\psi' + \alpha\beta e^{\beta\psi}{}_{\psi}'^{2} + \xi\psi' = 0 \qquad (6.8)$$

Making another transformation

$$\left.\begin{aligned} x &= \frac{\xi}{\sqrt{\alpha}} \\ y &= e^{\beta\psi} \quad \text{or} \quad \psi = \frac{1}{\beta}\,\ell n\, y \end{aligned}\right\} \qquad (6.9)$$

Then Equation (6.8) can be written

$$\alpha y\psi'' + \frac{\alpha}{\xi}\,y\psi' + \alpha\beta y\psi'^{2} + \xi\psi' = 0 \qquad (6.10)$$

$$\frac{d\psi}{d\xi} = \frac{d(\frac{1}{\beta}\,\ell n\, y)}{dx}\cdot\frac{dx}{d\xi} = \frac{1}{\beta}\cdot\frac{y'}{y}\cdot\frac{1}{\sqrt{\alpha}}$$

$$\frac{d^2\psi}{d\xi^2} = \frac{1}{\beta\sqrt{\alpha}}\cdot\frac{d(\frac{y'}{y})}{dx}\cdot\frac{dx}{d\xi} = \frac{1}{\beta\alpha}\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right)$$

Substituting these derivatives back into Equation (6.10)

$$\frac{\alpha y}{\beta\alpha}\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right) + \frac{\alpha}{x\sqrt{\alpha}}\cdot y\cdot\frac{1}{\beta\sqrt{\alpha}}\cdot\frac{y'}{y} + \frac{\alpha\beta y}{\beta^2\alpha}\cdot\frac{y'^2}{y^2} + \frac{x\sqrt{\alpha}}{\beta\sqrt{\alpha}}\cdot\frac{y'}{y} = 0$$

or
$$y'' - \frac{y'^2}{y} + \frac{y'}{x} + \frac{y'^2}{y} + \frac{xy'}{y} = 0$$

Finally,

$$y'' = -y'\left(\frac{1}{x} + \frac{x}{y}\right) \qquad (6.11.a)$$

with the transformed boundary conditions,

$$\left.\begin{aligned} x = x_o = \frac{r_o}{\sqrt{2\alpha t_o}}, \quad y = e^{\beta\theta_{sat}} \\ x = \infty \quad , \quad y = e^{\beta\theta_n} \end{aligned}\right\} \qquad (6.11.b)$$

The above equation is an ordinary differential equation that is nonlinear and it cannot be solved analytically. The qualitative properties of this equation are discussed in detail below before attempting to solve it.

The equation has two singularities, i.e., at $x = 0$ and $y = 0$ when the right side becomes undefined at these two points. Fortunately, x never vanishes for finite values of radius and time; and also the values of y are restricted between $e^{\beta\theta_n}$ and $e^{\beta\theta_{sat}}$ with no zero of y within this range of values. In the circumstances, these two singularities are automatically avoided. Another problem arises at $t = 0$, when x becomes undefined as seen from Equation (6.11.b). In this case also, the situation can be managed. At $t = 0$, it is assumed that the moisture in the soil is everywhere constant. A finite but small initial t can be taken. At this time, t_o, the moisture profile is not needed for the solution except for the two end points, which are already available as the boundary conditions.

Again looking at Equation (6.11.a), it is a boundary value problem with one condition at infinity. Boundary value problems are harder to solve numerically than initial value problems; but in the present case, one of the boundary conditions is unusual. The condition at

infinity is very hard to satisfy because infinity is beyond the thinking of man and computer both.

To get out of this situation, a study of the physical part of the problem is wise. This equation represents a physical situation, hence a solution is possible. It is known that at a finite time t_1, the wetting front would have moved to only a finite distance r_1. Beyond r_1 at this time, the moisture content of the soil is constant and equal to θ_n out to infinity. For the finite values of t_1 and r_1, the magnitude of x_1 is also finite by the relation,

$$x = \frac{r}{\sqrt{2\alpha t}} \tag{6.12}$$

As seen from Equation (6.12), the value of x can be increased even to infinity by making r equal to infinity at the fixed time t_1, but the value of y does not change and it remains constant at $e^{\beta\theta_n}$ beyond x_1. This means that the solution does not have to be carried out to an infinite value of x, but only up to $x = x_1$ where y becomes constant and does not change as x increases. With this argument it is apparent that

$$\left.\begin{aligned} x = x_o = \frac{r_o}{\sqrt{2\alpha t_o}} \quad , \quad y = e^{\beta\theta_{sat}} \\ x = x_1 = \frac{r_1}{\sqrt{2\alpha t_1}} \quad , \quad y = e^{\beta\theta_n} \end{aligned}\right\} \tag{6.11.b}$$

In Equation (6.11.b), x_o is known or given, but x_1 is not known, because the distance to the wetting front r_1 at time t_1 can

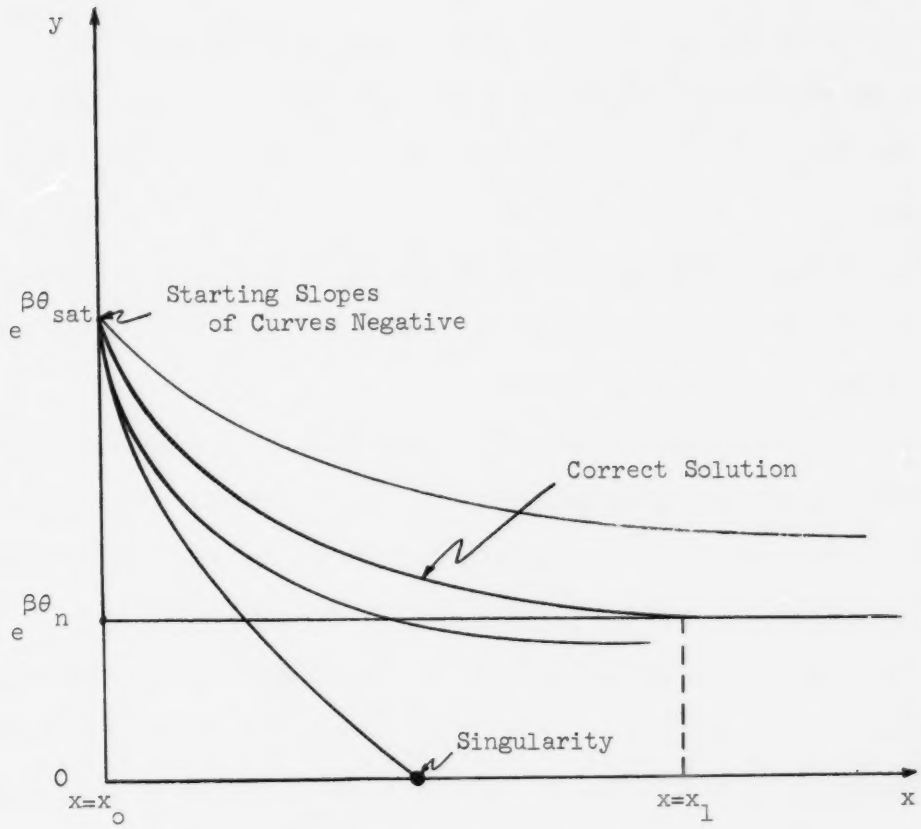
only be obtained from the solution of the equation. It means that sufficient conditions for the solution of the problem are not available, and some more supporting conditions must be found to solve the equation. This deficiency is removed by introducing the condition, arising out of the property of the solution, i.e.,

$$y'(x_1) = 0 \tag{6.13}$$

The above physical interpretation can also be inferred from the study of Equation (6.11.a). At $x = x_o$, x and y are positive, then it is evident from the equation that for negative values of y' at x_o, y'' is positive. It means that a curve of y starts turning in the counterclockwise direction as x is increased as may be seen in Figure 6.2.

The y-curves will continue to turn counterclockwise and the values of y' may become positive or the y-curves may intersect the $y = 0$ line. This latter possibility must be avoided, because the equation is undefined at $y = 0$. This situation, as evident from Figure 6.2, can be avoided by choosing larger values of y' than those which lead to singular points.

It is obvious now that y-curves (excluding the curves which meet $y = 0$ line) continuously turn counterclockwise and thus the value of y' increases. At some point x, y' will attain the value of zero, and physically it is known that the value of y' will continue to be zero henceforth without any change in the value of y. The same interpretation can be deduced from the equation also.



$$x = \frac{r}{\sqrt{2\alpha t}}$$

$$y = e^{\beta\theta}$$

Fig. 6.2: y-curves with Different Initial Values of y'.

As seen in Equation (6.11.a), y'' is zero if y' is zero. Upon integrating $y'' = 0$ one gets $y' = \text{constant}$, but y' is zero, hence the constant is zero, and thus from the equation also, it is found that y remains constant once y' becomes zero.

Now it is obvious that the actual solution of the Equation (6.11) is the one represented by that y-curve which finally becomes asymptotic with the $y(x_1) = e^{\beta\theta_n}$ line. To get this solution, two things must be done:

1. A good method of integrating the equation must be devised; and
2. The desired value of $y'(x_o)$ must be found by trial and error and interpolation.

The detail of the solution of the equation will be presented in the next chapter, but an approximate solution of Equation (6.11) can be found in combination with experiments in some circumstances as mentioned below.

It is well-known that any curve can be represented by a polynomial. For good accuracy, large numbers of points have to be taken on a curve and this makes the polynomial of very high degree; and it is not practicable to use such polynomials everywhere.

The solution of Equation (6.11.a), however, can be roughly represented by a 4^{th} degree polynomial. Before finding the polynomial solution of the equation, it is easier to transform the Equation (6.11) as follows:

$$\left.\begin{aligned} F(s) &= y \\ s &= \frac{x-x_o}{x_1-x_o} \end{aligned}\right\} \qquad (6.14)$$

$$y' = \frac{dF(s)}{ds} \cdot \frac{ds}{dx} = \frac{F'(s)}{x_1 - x_o}$$

Similarly

$$y'' = \frac{F''(s)}{(x_1 - x_o)^2} \quad \text{where} \quad F' = \frac{dF(s)}{ds} \quad .$$

Substituting back into Equation (6.11), the following set of equations are obtained.

$$\frac{F''}{(x_1 - x_o)^2} = - \frac{F'}{x_1 - x_o} \left(\frac{1}{s(x_1 - x_o) + x_o} + \frac{s(x_1 - x_o) + x_o}{F} \right) \qquad (6.15.a)$$

with boundary conditions,

$$\left.\begin{aligned} s = 0 \ , \quad F = e^{\beta\theta_s} \\ s = 1 \ , \quad F = e^{\beta\theta_n} \end{aligned}\right\} \qquad (6.15.b)$$

In Equation (6.15.a), x_1 and x_o are constant for a given soil. Now make the assumption that

$$F(s) = A + BS + CS^2 + DS^3 + ES^4 \qquad (6.16)$$

where A, B, C, D and E are constants.

Equation (6.16) contains five constants and five conditions are needed to evaluate them. Two conditions are available from the boundary conditions (Equation 6.15.b). Three more conditions are obtained from

the knowledge that at $s = 1$, the solution becomes constant and equal to $e^{\beta\theta_n}$ and all derivatives vanish. Hence

$$\text{At} \qquad s = 1\ ,\ F' = 0\ ,\ F'' = 0\ ,\ F''' = 0 \tag{6.17}$$

Using the boundary conditions in Equation (6.16), the following linear system of equations are obtained,

$$e^{\beta\theta_{sat}} = A \qquad [\text{from } 1^{st} \text{ of Eq. } (6.15.b)]$$

$$e^{\beta\theta_n} = e^{\beta\theta_{sat}} + B + C + D + E$$

$$0 = 0 + B + 2C + 3D + 4E$$

$$0 = 0 + 0 + 2C + 6D + 12E$$

$$0 = 0 + 0 + 0 + 6D + 24E$$

Solution of the above simultaneous equations gives

$$\left.\begin{aligned} A &= e^{\beta\theta_{sat}} \\ B &= 4(e^{\beta\theta_n} - e^{\beta\theta_{sat}}) \\ C &= 6(e^{\beta\theta_{sat}} - e^{\beta\theta_n}) \\ D &= 4(e^{\beta\theta_n} - e^{\beta\theta_{sat}}) \\ E &= e^{\beta\theta_{sat}} - e^{\beta\theta_n} \end{aligned}\right\} \tag{6.18}$$

This solution suggests a quick approximation of the problem, but it is not usable, because s depends on x_1, and x_1 is an unknown. If from experiment, the distance of the wetting front r_1 at time t_1 is known, then the solution is completely described.

It should be noted that this polynomial solution was found only from the end conditions and the known properties of the solution of the differential equation; but no use was made of the differential equation, hence it is only a very rough approximation. Improvement may be made in this procedure by substituting the polynomial solution into the Differential Equation (6.15) and determining the relation between r_1 and t_1 directly from the numerical solution of the equation rather than from experiment. In this circumstance the differential equation is still nonlinear and very hard to solve. This is a good area for future research to improve the polynomial solution.

Equation (6.11) can be written in a simpler form by one more transformation.

$$\left.\begin{aligned} w(z) &= y(x) \\ z &= \ell n\ x \end{aligned}\right\} \qquad (6.19)$$

$$y'(x) = \frac{dw(z)}{dz} \cdot \frac{dz}{dx} = \frac{w'(z)}{x}$$

$$y''(x) = \frac{d\left(w'(z)/x\right)}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\left[\frac{w''(z)}{x} - \frac{w'(z)}{x^2} \cdot x\right]$$

Substituting back into Equation (6.11)

$$\frac{w''}{x^2} - \frac{w'}{x^2} = -\frac{w'}{x}\left[\frac{1}{e^z} + \frac{e^z}{w}\right]$$

or

$$\frac{w''}{e^z} = -\frac{w'e^z}{w}$$

$$\therefore \qquad w'' = -\frac{e^{2z}}{w} \cdot w' \tag{6.20.a}$$

with conditions

$$\left.\begin{aligned} z_o &= \ell n(x_o) = \ell n\left(\frac{r_o}{\sqrt{2\alpha t_o}}\right), \quad w = \beta\theta_{sat} \\ z_1 &= \ell n(x_1) = \ell n\left(\frac{r_1}{\sqrt{2\alpha t_1}}\right), \quad w = \beta\theta_o \end{aligned}\right\} \tag{6.20.b}$$

Equation (6.20.a) is another form of Equation (6.11.a), and either one of them can be used for the solution of the two dimensional problem depending on the numerical technique used. Some further comments about Equation (6.20) will be made in the next chapter.

It is possible to find a few more transformations for Equation (6.20.a), and get rid of the independent variable completely, still the nonlinearity of the equation remains unsurmountable, and no analytic solution can be found at this stage of our knowledge of mathematics. The writer tried many such transformations without success. The methods used to solve Equation (6.11) is presented in the next chapter.

CHAPTER 7

APPROACH TO THE SOLUTION OF THE DIFFUSION EQUATION

The two dimensional diffusion equation without gravity has been transformed to a nonlinear, second-order, ordinary differential equation in Chapter 6. To solve the transformed ordinary differential Equation (6.11), some numerical techniques are needed, which are described below. These techniques are first developed for 1st order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x,y) \tag{7.1}$$

with an initial condition $y(x_o)$ = Constant C

Later, these techniques will be extended to 2nd-order ordinary differential equations, and the detail of successive applications of them in the present problem will be described.

The Runge-Kutta method: This method was developed by Kutta (1901) to solve 1st order ordinary differential Equation (7.1). This method is a modification of the Taylor series method, but the advantage of the Runge-Kutta method is that it does not require derivatives of $f(x,y)$. According to the Taylor series (taking up to 2nd-order terms) and using Equation (7.1),

$$y_{n+1} = y_n + f_n(\Delta x) + \left[\frac{\partial f_n}{\partial x} + \frac{\partial f_n}{\partial y} \cdot f_n \right] \frac{(\Delta x)^2}{2} + \cdots \tag{7.2}$$

where $y_{n+1} = y(x_{n+1})$ as shown in Figure 7.1,

$f_n = f(x_n, y_n)$, and

n denotes the number of x intervals.

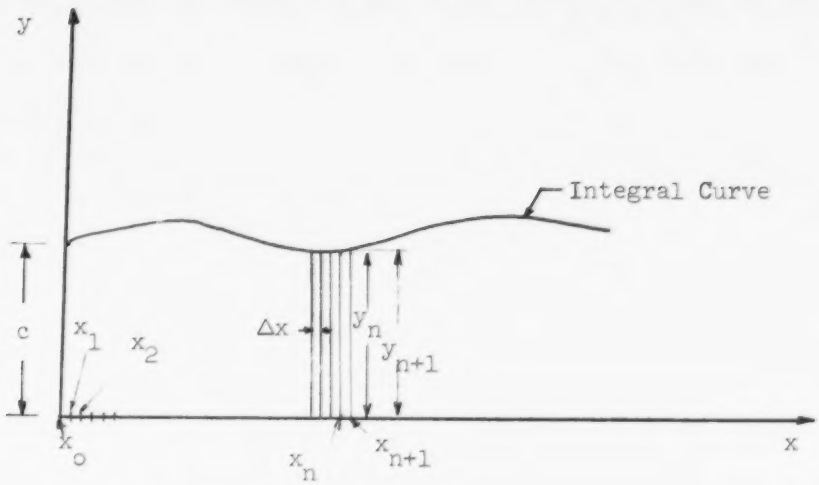


Fig. 7.1: Step by Step Integration of 1st Order Ordinary Differential Equation.

Assuming that Equation (7.2) is equivalent to

$$y_{n+1} = y_n + a \cdot \Delta x \cdot f_n + b \cdot \Delta x \cdot f(x_n + c \cdot \Delta x,\ y_n + e \cdot \Delta x \cdot f_n) \quad (7.3)$$

where a, b, c and e are constants.

Constants in Equation (7.3) are to be determined in such a way that this equation is identical to Equation (7.2). Writing the Taylor series expansion of f(x,y),

$$f(x+\Delta x,\ y+\Delta y) = f(x,y) + \Delta x \cdot \frac{\partial f(x,y)}{\partial x} + \Delta y \cdot \frac{\partial f(x,y)}{\partial y} + \cdots$$

$$\therefore\ f(x_n+c\cdot\Delta x,\ y_n+e\cdot\Delta x\cdot f_n) = f(x_n,y_n) + c \cdot \Delta x \cdot \frac{\partial f(x_n,y_n)}{\partial x}$$

$$+ e \cdot \Delta x \cdot \frac{\partial f(x_n,y_n)}{\partial y} f_n + \cdots$$

$$= f_n + \Delta x \left[c \frac{\partial f_n}{\partial x} + e \cdot \frac{\partial f_n}{\partial y} \cdot f_n \right] + \cdots \qquad (7.4)$$

Substituting Equation (7.4) into Equation (7.3)

$$y_{n+1} = y_n + (a+b)f_n \cdot \Delta x + b(c \frac{\partial f_n}{\partial x} + e \cdot \frac{\partial f_n}{\partial y} \cdot f_n)(\Delta x)^2 + \cdots \qquad (7.5)$$

To make Equations (7.2) and (7.5) to agree, the following must hold

$$\left.\begin{aligned} a + b &= 1 \\ c = e &= \frac{1}{2(1-a)} \end{aligned}\right\} \qquad (7.6)$$

Any combination of values prescribed to constants, satisfying Equations (7.6) will serve the purpose. Usually a convenient and symmetric set of values are chosen. So making a choice

$$a = b = \frac{1}{2}$$

$$c = e = 1$$

Equation (7.3) becomes

$$y_{n+1} \approx y_n + \frac{\Delta x}{2} \left[f(x_n,y_n) + f(x_n+\Delta x,\ y_n+\Delta x f_n) \right] \qquad (7.7)$$

Equation (7.7) can be used for the integration of 1st order differential equations, but it is customarily written in a slightly different form. Let

$$\left.\begin{aligned} K_1 &= \Delta x \cdot f(x_n, y_n) \\ \text{and} \quad K_2 &= \Delta x \cdot f(x_n + \Delta x,\ y_n + K_1) \quad , \end{aligned}\right\} \tag{7.8a}$$

then Equation (7.7) reduces to

$$y_{n+1} \approx y_n + \frac{1}{2}(K_1 + K_2) \tag{7.8b}$$

This is the second order Runge-Kutta method. It should be remembered that the choice of constants is not unique, and other choices are possible except a or b should not equal zero. When a differential equation is integrated step-by-step using Equation (7.8), then error is of the order $(\Delta x)^3$, because the equation is equivalent to a Taylor series up to second derivatives.

Similarly the fourth order Runge-Kutta formula can be obtained. The derivation of 4th order equation is omitted here, because it is extremely involved and very lengthy. The 4th order Runge-Kutta formula is as follows:

$$y_{n+1} \approx y_n + \frac{1}{6}(a_1 + 2a_2 + 2a_3 + a_4) \tag{7.9}$$

where

$$\left.\begin{aligned} a_1 &= \Delta x \cdot f(x_n,\ y_n) \\ a_2 &= \Delta x \cdot f\left(x_n + \frac{\Delta x}{2},\ y_n + \frac{a_1}{2}\right) \\ a_3 &= \Delta x \cdot f\left(x_n + \frac{\Delta x}{2},\ y_n + \frac{a_2}{2}\right) \\ a_4 &= \Delta x \cdot f(x_n + \Delta x,\ y_n + a_3) \end{aligned}\right\} \tag{7.10}$$

The error bound in Equation (7.10) is of the order $(\Delta x)^5$. The advantage of the Runge-Kutta method is its simplicity, and step-by-step calculation can be carried on with only one set of previous values of x and y. Equation (7.10) is only applicable to 1st order ordinary differential equations, but its usefulness can be extended to a second order ordinary differential equation also.

Taking the second order ordinary differential equation

$$\frac{d^2y}{dx^2} = F(x,y, \frac{dy}{dx}) \tag{7.11}$$

with two given initial conditions

$y(x_o) = C_1$ and $y'(x_o) = C_2$, where C_1 and C_2 are constants, this equation can be written as a system of two simultaneous 1st order ordinary differential equations as follows:

$$\left.\begin{aligned} g = \frac{dy}{dx} &= f(x,y) \\ \frac{dg}{dx} &= F(x,y,g) \end{aligned}\right\} \tag{7.12}$$

with initial conditions

$$y(x_o) = C_1 \quad \text{and} \quad g(x_o) = C_2 \ .$$

Equation (7.12) can be written in the fourth order Runge-Kutta formula as

$$\left.\begin{aligned} y_{n+1} &\approx y_n + \frac{1}{6}(a_1 + 2a_2 + 2a_3 + a_4) \\ g_{n+1} &\approx g_n + \frac{1}{6}(b_1 + 2b_2 + 2b_3 + b_4) \end{aligned}\right\} \qquad (7.13)$$

where

$$\left.\begin{aligned} a_1 &= \Delta x \cdot f(x_n, y_n) \\ a_2 &= \Delta x \cdot f\left(x_n + \frac{\Delta x}{2}, y_n + \frac{a_1}{2}\right) \\ a_3 &= \Delta x \cdot f\left(x_n + \frac{\Delta x}{2}, y_n + \frac{a_2}{2}\right) \\ a_4 &= \Delta x \cdot f(x_n + \Delta x, y_n + a_3) \end{aligned}\right\} \qquad (7.14a)$$

and

$$\left.\begin{aligned} b_1 &= \Delta x \cdot F(x_n, y_n, g_n) \\ b_2 &= \Delta x \cdot F\left(x_n + \frac{\Delta x}{2}, y_n + \frac{a_1}{2}, g_n + \frac{b_1}{2}\right) \\ b_3 &= \Delta x \cdot F\left(x_n + \frac{\Delta x}{2}, y_n + \frac{a_2}{2}, g_n + \frac{b_2}{2}\right) \\ b_4 &= \Delta x \cdot F(x_n + \Delta x, y_n + a_3, g_n + b_3) \end{aligned}\right\} \qquad (7.14b)$$

Substituting Equation (7.12) into Equation (7.14.a), the following are obtained,

$$\left.\begin{aligned} a_1 &= \Delta x \cdot y'_n \\ a_2 &= \Delta x \cdot y'_n + b_1 \cdot \frac{\Delta x}{2} \\ a_3 &= \Delta x \cdot y'_n + b_2 \cdot \frac{\Delta x}{2} \\ a_4 &= \Delta x \cdot y'_n + b_3 \cdot \Delta x \end{aligned}\right\} \qquad (7.15)$$

Inserting Equation (7.15) into Equation (7.13), gives

$$\left.\begin{aligned} y_{n+1} &\approx y_n + \Delta x \cdot y'_n + \frac{\Delta x}{6}\,(b_1+b_2+b_3) \\ y'_{n+1} &\approx y'_n + \frac{1}{6}\,(b_1 + 2b_2 + 2b_3 + b_4) \end{aligned}\right\} \qquad (7.16.a)$$

where

$$\left.\begin{aligned} b_1 &= \Delta x \cdot F(x_n,\ y_n,\ y'_n)\ , \\ b_2 &= \Delta x \cdot F\left(x_n + \frac{\Delta x}{2},\ y_n + \frac{\Delta x}{2} \cdot y'_n,\ y'_n + \frac{b_1}{2}\right)\ , \\ b_3 &= \Delta x \cdot F\left(x_n + \frac{\Delta x}{2},\ y_n + \frac{\Delta x}{2}\, y'_n + \frac{\Delta x}{4} \cdot b_1,\ y'_n + \frac{b_2}{2}\right)\ , \\ b_4 &= \Delta x \cdot F\left(x_n + \Delta x,\ y_n + \Delta x \cdot y'_n + \frac{\Delta x}{2} \cdot b_2,\ y'_n + b_3\right)\ . \end{aligned}\right\} \qquad (7.16.b)$$

The writer has applied Equation (7.16) for the solution of Equation (6.11) for the first three steps, because the accuracy of this method is limited to 4th order. The rest of the steps have been carried on with the Adam's Predictor-Corrector method. Adam's method needs four initial points along the integration curve, hence to get these points, first three steps were calculated by the Runge-Kutta method.

Adams method: If a function $f(x)$ is given on $k + 1$ equally spaced points on the x-axis, then the values of these points can be denoted by

$$x_n,\ x_{n-1} = x_n - \Delta x, \ldots\ldots, x_{n-k} = x_n - k \cdot \Delta x\ .$$

With the notation $f(x_n) = f_n$, already used, the forward differences are written as

$$\Delta f_m = f_{m+1} - f_m;\ \Delta^2 f_m = \Delta f_{m+1} - \Delta f_m = f_{m+2} - 2f_{m+1} + f_m\ ;$$

and so on. Similarly the backward differences are written as

$$\nabla f_m = f_m - f_{m-1}; \quad \nabla^2 f_m = \nabla f_m - \nabla f_{m-1} = f_m - 2f_{m-1} + f_{m-2} \quad ;$$

and so on.

Using the backward difference notations, Newton's formula for a polynomial approximation of a function $f(x)$ is given by,

$$f(x_n + r\Delta x) \approx f_n + r \cdot \nabla f_n + \frac{r(r+1)}{2!} \nabla^2 f_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_n + \cdots \cdots + \frac{r(r+1)(r+2) \cdots (r+k-1)}{k!} \nabla^k f_n \qquad (7.17)$$

where $x = x_n + r \cdot \Delta x$ or $r = \dfrac{x - x_n}{\Delta x}$,

and total length of interval is $k \cdot \Delta x$.

The above polynomial is also known as Gregory's polynomial and it interpolates a smooth function $f(x)$ at any point of the interval. This polynomial gives exact values of $f(x)$ at $k + 1$ points at intervals of Δx over the range. It can also be used for extrapolation. The derivation of this polynomial can be found in suitable references given in the bibliography.

For a first-order differential Equation (7.1), the integration can be carried out with this polynomial on an interval Δx. Noting that

$$\int_{x_n}^{x_n + \Delta x} f(x)dx = \Delta x \int_0^1 f(x_n + r \cdot \Delta x)dr \quad ,$$

both sides of Equation (7.1) can be integrated as,

$$\int_{x_n}^{x_n+\Delta x} dy = \int_{x_n}^{x_n+\Delta x} f(x,y(x))dx \approx \Delta x\Big[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \cdots\Big]$$

$$\therefore\ y_{n+1} \approx y_n + \Delta x\Big[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \cdots\Big] \tag{7.18}$$

Equation (7.18) is the Adams Predictor formula of 4th order for integrating 1st order ordinary differential equations. This equation can be written in an explicit form as follows:

$$y^p_{n+1} = y_n + \frac{\Delta x}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] \tag{7.19}$$

As seen in Equation (7.19), four values of y at four previous points of x are needed to obtain an advance value of y. This formula predicts a new value of y from four known values of y, hence a superscript P has been used with y.

Similarly a corrector formula is obtained by integrating the 1st order ordinary differential equation over the interval x_{n-1} to x_n with the help of Newton's polynomial. Integrating Equation (7.1)

$$\int_{x_{n-1}}^{x_n} dy = \int_{x_{n-1}}^{x_n} f(x,y(x))dx \approx \Delta x\Big[f_n - \frac{1}{2}\nabla f_n - \frac{1}{12}\nabla^2 f_n - \frac{1}{24}\nabla^3 f_n - \frac{19}{720}\nabla^4 f_n + \cdots\Big] \tag{7.20}$$

Equation (7.20) is obviously better than Equation (7.19), because the coefficients on the right side of Equation (7.20) decrease faster and become negligible sooner than in the right side of Equation (7.19). The difficulty with Equation (7.20) is that it contains y_n which is intended to be determined from the solution of the equation. This difficulty is overcome by first getting a predicted value of y^p with the use of Equation (7.19) and correcting it by applying Equation (7.20). Equation (7.20) can be applied as many times as desired until the values of y^p and y^c completely converge. This method is extremely convergent and only one or two applications of the corrector formula may be sufficient in most of the cases. The corrector formula can be written explicitly as,

$$y^c_{n+1} \approx y_n + \frac{\Delta x}{24}\left[9f_{n+1}+19f_n-5f_{n-1}+f_{n-2}\right] \qquad (7.21)$$

The error bound in the case of the 4th order Adam's Predictor-Corrector method is given by

$$E_{n+1} \approx \frac{y^p_{n+1} - y^c_{n+1}}{18} \qquad (7.22)$$

where E_{n+1} is error bound at $n+1$ steps.

The Predictor-Corrector method can be applied to second order differential equations also by writing a second order equation as two 1st order equations as shown in Equations (7.11) and (7.12). Then these two equations can be solved one at a time as 1st order differential equations by applying these formulas.

Method of Solution

The two methods mentioned above have been used to solve the transformed ordinary differential equation. Restating Equations (6.11),

$$y'' = -y'\left(\frac{1}{x} + \frac{x}{y}\right) \tag{7.23.a}$$

with the boundary conditions

$$\left.\begin{aligned} x = x_o \;,\quad y = e^{\beta\theta_{sat}} \\ x = x_1 \;,\quad y = e^{\beta\theta_n} \end{aligned}\right\} \tag{7.23.b}$$

To solve Equation (7.23), first appropriate values of x_o and Δx are selected. An arbitrary value of $y'(x_o)$ is assumed so that the equation becomes an initial value problem. To start step-by-step solution, Equation (7.16) is applied to Equation (7.23) three times. In this way three more values of y corresponding to values of x are obtained; and with the initial conditions, four beginning points are known. Now the Adam Predictor-Corrector method is applied. For this, Equation (7.23) is written as two first order equations

$$\left.\begin{aligned} y' &= g \\ g' &= -g\left(\frac{1}{x} + \frac{x}{y}\right) \end{aligned}\right\} \tag{7.24}$$

with the initial conditions

$$y(x_o) = e^{\beta\theta_{sat}} \;,\quad y'(x_o) = \text{assumed} \;.$$

Then the Predictor formula (7.19) is applied successively to

both Equations of (7.24). This gives new values of y^p and y'^p corresponding to the next interval of x. With these predicted values, the Corrector formula (7.21) is applied to get somewhat modified values y^c and y'^c at this interval of x.

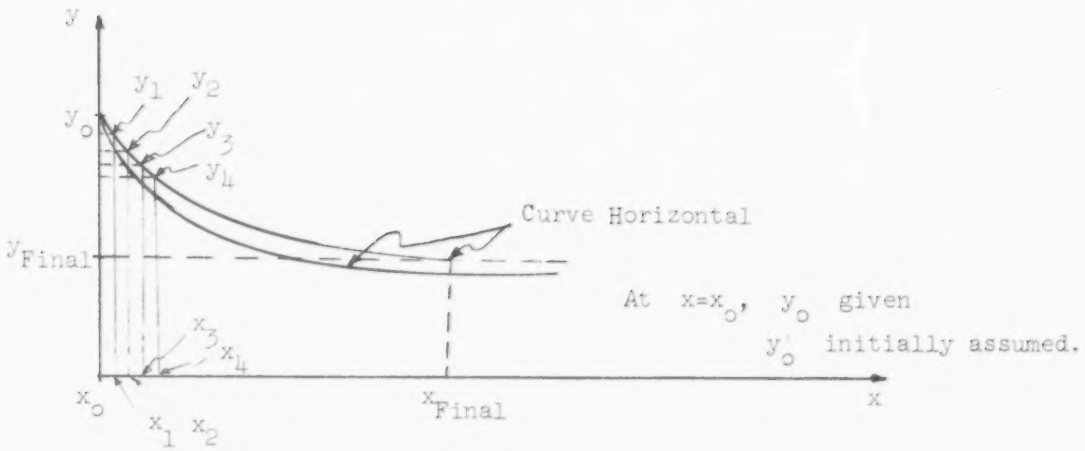


Fig. 7.2: Integration of Differential Equation.

As shown in Figure 7.2, starting with $y(x_0)$ and assumed values of $y'(x_0)$; $y(x_1)$, $y(x_2)$ and $y(x_3)$ and their first derivatives are obtained by solving the differential equations with the help of Equation (7.16). With four values of y and y' corresponding to four points on x-axis (x_0, x_1, x_2 and x_3) available, the Predictor formula is used to get $y(x_4)$ and $y'(x_4)$. Taking $y^p(x_4)$ and $y'^p(x_4)$ as approximations for y_4 and y'_4, they are used in the Corrector formula to get improved values of y_4 and y'_4 as $y^c(x_4)$ and $y'^c(x_4)$.

The Corrector formula is applied twice for better approximation of y at each step. The error bound is calculated at each step with Equation (7.22), and with the double application of the Corrector formula, it is found to be of order 10^{-7} in the present case. For still more accurate results, the Corrector formula can be applied a few more times until the convergence is perfect within the limits of decimal points given by the computer. The writer thinks one or two more applications of the Corrector formula will reduce the error bound below 10^{-12} (beyond the single accuracy capacity of the Burroughs 5500), and not much additional computer time will be required for this. Provision has been made in the computer program for this possibility.

The above method of integrating the differential equation is carried on until y becomes constant and y' becomes zero. Then a comparison is made between the y's, obtained from the solution of the differential equation and the desired value of $y = e^{\beta\theta_n}$. If they are not equal, then a new value of y'_o is guessed and the solution of the differential equation is started all over again. Similarly in the beginning three guessed values of y'_o are tried and three different solutions of the differential equation are obtained. If none of these three solutions are the desired one, then a polynomial curve is fitted with these three solutions relating y_{final} and $y'(x_o)$ as shown in Figures 7.3 and 7.4.

The Lagrangian polynomial formula is used for the curve fitting as shown in Figure 7.4. The Lagrangian polynomial to approximate a function $f(x)$ is given below, but its derivation is omitted as it may be found in the references listed at the end of dissertation.

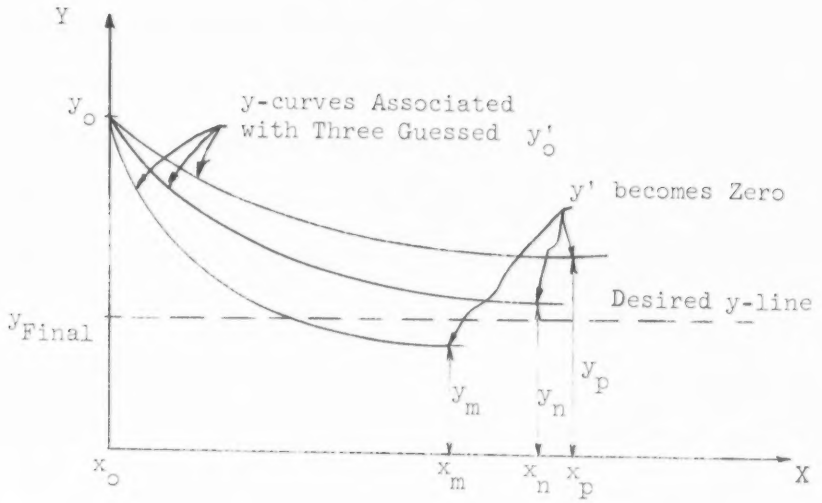


Fig. 7.3: Solutions with Guessed $y'(x_o)$.

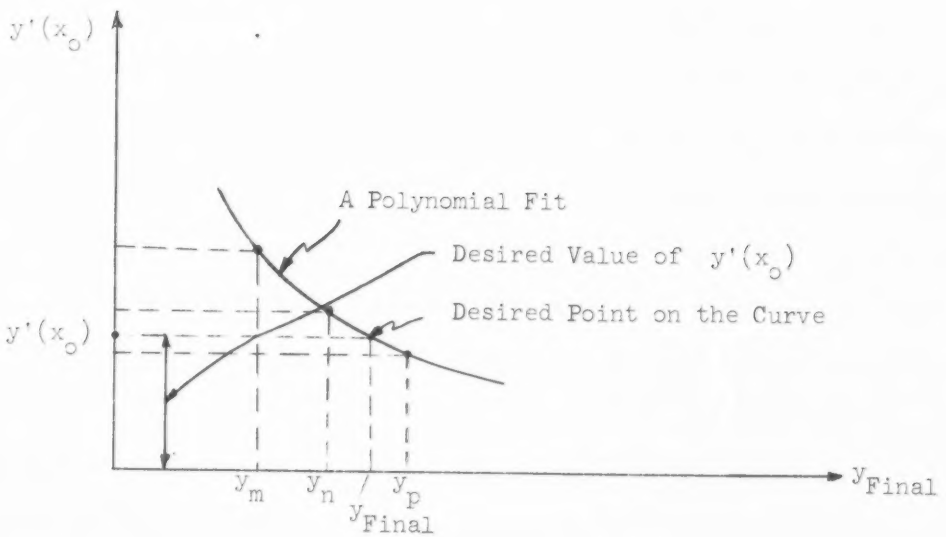


Fig. 7.4: Interpolation for $y'(x_o)$.

$$f(x) \approx \sum_{k=0}^{n} L_k(x)\, f(x_k) \qquad (7.25.a)$$

where

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x-x_j}{x_k-x_j}$$

$$= \frac{(x-x_o) \cdots (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_n)}{(x_k-x_o) \cdots (x_k-x_{k-1})(x_k-x_{k+1}) \cdots (x_k-x_n)} \qquad (7.25.b)$$

Three values of y obtained from the solution of the differential equation are substituted into Equation (7.25) as independent variables and corresponding values of y'_o are treated as dependent variables $y'_o(y)$ and a polynomial is computed passing through these three points. Equation (7.25) gives a polynomial of nth degree if $n+1$ values of a function $f(x)$ are taken. For the three points, the polynomial will be of second degree. After having a polynomial approximating $y'(y)$, the exact value of y_{final} is substituted into it to get a corresponding value of y'_o. With this computed value of y'_o, the differential equation is again solved as usual and a comparison is made between y obtained from the solution and y_{final}. If the solution is not within the desired accuracy, then with the four values, the described polynomial interpolation is made and this is continued until the desired solution of the differential equation is obtained within the permissible accuracy. The above mentioned methods seem to be complicated, but they are efficient and accurate on the digital computer and the writer has developed and run a program on the Burroughs 5500, which will be

presented in Appendix 1.

An attempt was made to solve Equation (6.20) by the above methods for the case of Yolo Light Clay, but very large initial values of w' had to be assumed to start the solution. Under such circumstances, the values of w change rapidly even for small intervals of z, and the solution showed instability. This equation however may be applicable to soils having properties different from the Yolo Light Clay.

Once the solution of the ordinary differential Equation 6.11 is obtained, the solution is transformed back to get the results in terms of r, t and θ. These transformations are accomplished with the following formulas and tabulated by the computer

$$\left.\begin{aligned} \theta &= \frac{1}{\beta} \ln y \\ r &= x \cdot \sqrt{2\alpha t} \end{aligned}\right\} \qquad (7.26)$$

Using Equations (7.26), θ and r are computed with the desired values of t. Two soil samples have been solved as examples by the above methods on the B 5500 computer which will be presented in later chapters.

Method of Accounting for Gravity

The solution described as yet is only good in the absence of gravity. An approximate method used by the writer to compute the influence of gravity is now explained. Rewriting the difference Equation (5.2),

$$\theta^{n+1}_{k,m}\left[-\frac{1}{\Delta t}-\frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}}{(\Delta r)^2}-\frac{D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2}-\frac{D^{n+\frac{1}{2}}_{k,m+\frac{1}{2}}}{r^2_{k,m}(\Delta\phi)^2}-\frac{D^{n+\frac{1}{2}}_{k,m-\frac{1}{2}}}{r^2_{k,m}(\Delta\phi)^2}\right]$$

$$+\theta^{n+1}_{k-1,m}\left[-\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r}+\frac{D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2}\right]+\theta^{n+1}_{k+1,m}\left[\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r}+\frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}}{(\Delta r)^2}\right]$$

$$+\theta^{n+1}_{k,m-1}\left[\frac{D^{n+1}_{k,m-\frac{1}{2}}}{r^2_{k,m}(\Delta\phi)^2}\right]+\theta^{n+1}_{k,m+1}\left[\frac{D^{n+\frac{1}{2}}_{k,m+\frac{1}{2}}}{r^2_{k,m}(\Delta\phi)^2}\right]+$$

$$\left[\frac{\theta^{n}_{k,m}}{\Delta t}-\sin\phi_{k,m}\;\frac{K^{n+\frac{1}{2}}_{k-1,m}-K^{n+\frac{1}{2}}_{k+1,m}}{2\Delta r}-\cos\phi_{k,m}\;\frac{K^{n+\frac{1}{2}}_{k,m-1}-K^{n+\frac{1}{2}}_{k,m+1}}{2r_{k,m}\cdot\Delta\phi}\right]=0 \qquad (7.27)$$

and making the assumptions that

$$\theta^{n+1}_{k,m}\approx\frac{\theta^{n+1}_{k,m-1}+\theta^{n+1}_{k,m+1}}{2}$$

and $K^{n+\frac{1}{2}}_{k,m-1}-K^{n+\frac{1}{2}}_{k,m+1}$ is negligible compared to

$K^{n+\frac{1}{2}}_{k-1,m}-K^{n+\frac{1}{2}}_{k+1,m}$, Equation (7.27) reduces to

$$\theta^{n+1}_{k,m}\left[-\frac{1}{\Delta t}-\frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}+D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2}\right]+\theta^{n+1}_{k-1,m}\left[-\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r}+\frac{D^{n+\frac{1}{2}}_{k-\frac{1}{2},m}}{(\Delta r)^2}\right]$$

$$+\theta^{n+1}_{k+1,m}\left[\frac{D^{n+\frac{1}{2}}_{k,m}}{2r_{k,m}\cdot\Delta r}+\frac{D^{n+\frac{1}{2}}_{k+\frac{1}{2},m}}{(\Delta r)^2}\right]+\left[\frac{\theta^{n}_{k,m}}{\Delta t}-\sin\phi_{k,m}\;\frac{K^{n+\frac{1}{2}}_{k-1,m}-K^{n+\frac{1}{2}}_{k+1,m}}{2\Delta r}\right]$$

$$=0 \qquad (7.28)$$

The assumptions made to approximate the difference equation are actually very reasonable. The change of θ in the Φ direction at a constant r due to gravity is small and is zero when gravity is not accounted for as seen from Figures 5.1 and 6.1. The corresponding change of θ in radial direction is large compared to the change in the direction of polar angle. Under the circumstances, the neglected or approximated terms are not noticeable in the solution and they may be neglected for all practical purposes.

Now studying Equation (7.28), it is obvious that the contributions of all terms have been accounted for in the solution of the symmetric problem except that of the last term in the last bracket. The contribution in the value of $\theta_{k,m}^{n+1}$ due to this term is

$$\Delta\theta_{k,m}^{n+1} = - \frac{\sin \Phi_{k,m} \left\{ \frac{K_{k-1,m}^{n+\frac{1}{2}} - K_{k+1,m}^{n+\frac{1}{2}}}{2\Delta r} \right\}}{\frac{1}{\Delta t} + \frac{D_{k+\frac{1}{2},m}^{n+\frac{1}{2}} + D_{k-\frac{1}{2},m}^{n+\frac{1}{2}}}{(\Delta r)^2}} \qquad (7.29)$$

The writer has used the names "the symmetric problem" and "the symmetric solution" for "the diffusion problem without gravity" and "its solution", and they should not be confused.

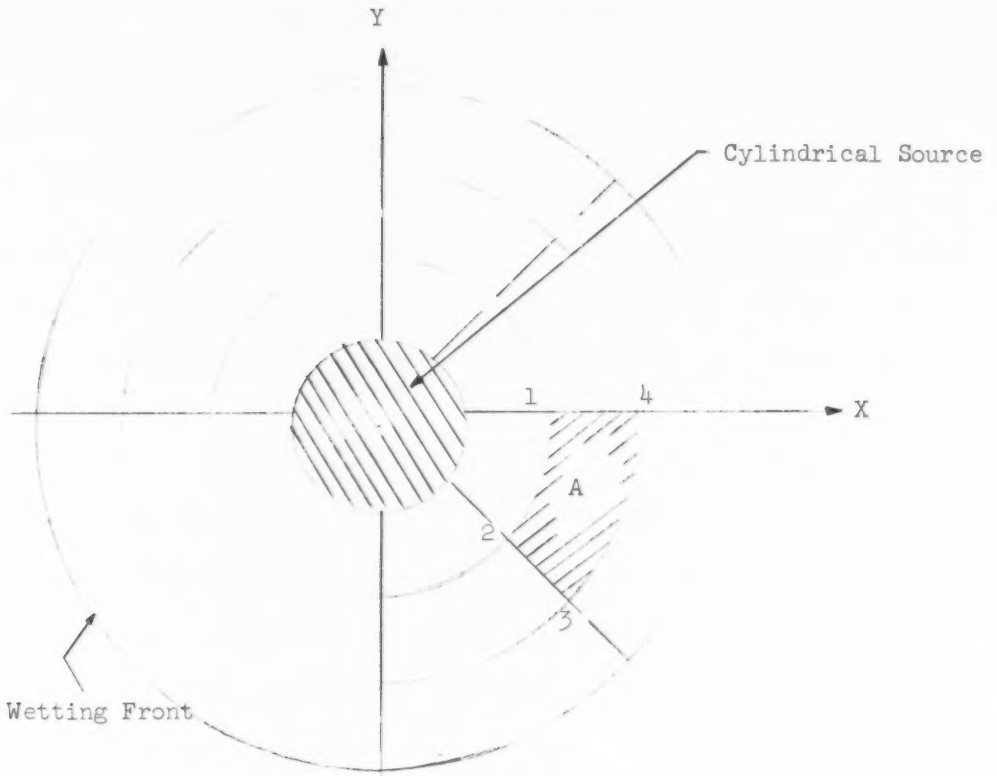
Once the solution for the symmetric problem is obtained, then a grid on the half plane is again made. With the known values of θ for each time interval, the contribution of gravity to each intersection of the mesh is calculated with Equation (7.29) and these contributions are added to the values of θ obtained from the symmetric solution.

These contributions to the values of θ are computed for each intersection on the grid and successively from one time interval to another, everytime accounting for the accumulative contribution to θ due to gravity. This method also has been programmed in Extended Algol Language, used for Burroughs 5500. If desired, the neglected terms may also be accounted for in this method and here is a good area for future research on the improvement of the gravity correction.

Computation of Infiltration Rate

Finally, the computation of the infiltration rate for this problem was also programmed. After getting the complete solution with gravity, the region between the cylindrical source and the wetting front at any time interval is suitably divided into many concentric areas as shown in Figure 7.5. Only half of the region on one side of the vertical axis is used for computation and the final value is doubled to get the total infiltration. Each of the concentric areas is also divided into smaller areas by dividing in the polar angular direction.

Water contained in each small element of area is computed separately as shown in Figure 7.5 and water content of all areas on half of the region are added together and doubled to get the total quantity of infiltration into an unit thickness of soil up to the time under consideration. If the moisture content is given on the weight basis, then the final quantity thus obtained must be multiplied by the specific gravity of the fully saturated soil to get the volume of total infiltration. In this way, the total infiltration volume for different time intervals is computed.



Total Water in Hatched Area

$$= A \times \left[\frac{\theta_1+\theta_2+\theta_3+\theta_4}{4} - \theta_n \right]$$

Fig. 7.5: Computation of Infiltration.

To obtain the infiltration rate at different times, the difference of total infiltration at two successive times is taken and divided by the time interval. In this way, many infiltration rates corresponding to different time intervals are calculated and plotted to see the variation of infiltration rate with time, which has been done in later chapters for the example chosen.

It should be remarked that the total infiltration computed for the symmetric solution without gravity and for the assymmetric solution with gravity at any time is nearly the same for all practical purposes. This is easily justified by looking at Figure 6.1. The unbalance caused by gravity due to the increase in the quantity of water in the lower half of the region is approximately offset by the decrease in the quantity of water in the upper half of the region at any time.

The above methods have been used to solve for the unsteady and unsaturated two dimensional flow from a cylindrical source for two soils; the details of the computations are presented in the next chapter.

CHAPTER 8

OPERATION OF THE COMPUTER PROGRAM AND EXAMPLES

The program in extended Algol language for the solution of unsteady and unsaturated flow in soils in two dimensions is presented in Appendix 1. This program can be used for any soil having an exponential diffusivity relationship; but for each particular soil, some of the assignment statements in the program have to be changed. In the program presented in Appendix 1, assignments are shown for Pachappa Loam with a radius of 1 cm. for the cylindrical source. For simplicity the use of data cards has been eliminated.

In order to be able to use the program for a soil, the values of $K(\theta)$ and $D(\theta)$ must be known from experiment. The relations for K and D should be fitted as given in Equations (3.5) and (3.10) respectively. However, experimental values of K may be used directly with some modification of the program, and thus the use of Equation (3.5) can be avoided. But the writer found Equation (3.5) very satisfactory for the two soils chosen for examples and use of the equation reduced the computer time.

In Table 8.1 is presented the detail of changes to be made in the program for any soil. Only those cards which are mentioned in the table are changed. The only changes made are in the numerical values as described in the table except for the comment cards.

The changes are of two types:

1. Mandatory: Cards which must be changed from one soil to another. The data punched on these cards represent

the physical characteristics of the soil.

2. Optional: Cards on which changes are not essential. However, when a higher degree of accuracy is desired, then changes are necessary.

A remark column is added in the table to describe the merits of changes, whenever needed.

TABLE 8.1

Card No.	Changes in the Program	Remarks
3	Appropriate soil type "Pachappa Loam" in the present case	For information only. No effect on the execution.
31	Value of "α"	Equation (3.10)
32	Value of "β"	Equation (3.10)
33	Value of "θ_{sat}"	
34	Value of "θ_n" (initial moisture content)	
35	Guessed values of "y'(0)". Initially 3	Trial is needed.
36	Values are guessed and assigned to these 3 cards. If they don't work, another set of values are guessed, until the program works.	
37		
38	Total number of integration-steps initially selected. This is known as II in the program.	

Table 8.1 (continued)

Card No.	Changes in the Program	Remarks
39	Selected step size. For Pachappa Loam, 0.1 has been chosen. This can be reduced for higher degree of accuracy.	Optional
40	Initial time t_o. Last number in the parenthesis is replaced by t_o. For Pachappa Loam, one minute was selected. Numerator 1 is the radius of cylindrical source. It can be changed as needed.	These can be chosen arbitrarily small, but greater than zero.
48	Minimum interval of steps, at which output of x and y is desired. Change both numbers on this card. Always integer and should divide II Perfectly.	Optional
71	Same as card no. 48.	Optional. If change is made, it should be same as card no. 48.
76	A convenient number by which II should be increased, in case integration has not become asymptotic as yet.	Optional
77	Same as card no. 76	Optional
82	Tolerance, can be changed as desired.	Optional
97	Comment should be changed when values of R and DT are changed.	For information only.
105	Value of K_{sat}.	
107	Change 40 to any number of time intervals	

Table 8.1 (continued)

Card No.	Changes in the Program	Remarks
	desired, R[I,0] should be assigned the value of the radius of cylindrical source.	
108	In this For-loop, initial value of I and its increment can be set to get as many points on the moisture profile as desired, but numbers should be adjusted so that (II - Initial I) is divisible by the increment of I. In the present case; II = 400, $I_{initial}$ = 50, increment = 50, hence (400/50)+1 = 9 points were printed on the moisutre profile including end points at each time interval.	
110	(II - $I_{initial}$)/increment as in card no. 108.	
113	In this card, Equation (6.5) is assigned to variable KP. Numbers on this card should be replaced by proper values of K_{sat}, θ_o, θ_o and n respectively for the soil. Numbers on the card are for Pachappa Loam.	
116	In this For-loop, desired initial and final values of time with its increments should be assigned.	

Table 8.1 (continued)

Card No.	Changes in the Program	Remarks
119	"1" in the "write" statement should be changed to the radius of the source.	No effect on the execution, but output will be wrong.
120	Same as card no. 108.	
131	Specific Gravity of saturated soil, in case θ is given on the weight-basis, otherwise one. When θ is given on the volume-basis, SG must be assigned unity otherwise infiltration volume will be computed wrong.	
133	Increment of time as in card no. 116.	
141	In place of @-1, the value of $(1/(2 \times \Delta t))$ should be assigned, where Δt is the increment of time as above. For Pachappa Loam, $\Delta t = 5$ min., hence @-1 was used on this card.	
155	The increment of time.	
160	Total number of time increments.	

If the number of time increments and integration steps is

large[1] the bounds of certain array declarations may have to be changed. For general problems, the range of array bounds given in the program is sufficient.

Two examples chosen by the writer are for Yolo Light Clay and Pachappa Loam. Some of the relationships for K and D for these soils have been plotted in Chapter 3. The following information was furnished for these soils:

Yolo Light Clay:[2] θ given on the weight-basis.

Specific Gravity = 1.32

$\theta_{sat} = 0.375$

$\theta_o = 0.18$

$\theta_n = 0.18$ (Assumed initial uniform moisture content of the soil)

$$K = 12.3 \times 10^{-6} \left(\frac{\theta - 0.18}{0.375 - 0.18} \right)^{4.5} \text{ cm/sec}$$

$$D = \frac{e^{32\theta}}{16 \times 10^6} \text{ cm}^2/\text{sec}$$

---

[1] If the number of integration steps is greater than 800, the array bounds of X, Y and DY on card no. 15 should be changed as desired. If the number of time intervals is greater than 40, the first array bound of R on card no. 95 should be changed. If the number of points on the moisture profiles including end points is greater than ten, then the second array bound of R on this card should be changed as needed.

[2] Moore, R. E., Water Conduction from Shallow Water Tables, Hilgardia, 12: 383-426 (1939).

Pachappa Loam:[3] θ given on the weight-basis.

Specific Gravity = 1.40

$\theta_{sat} = 0.33$

$\theta_o = 0.04$

$\theta_n = 0.10$ (Assumed initial uniform moisture content of the soil)

$$K = 6.25 \times 10^{-3} \left(\frac{\theta - 0.04}{0.33 - 0.04} \right)^6 \text{ cm/min.}$$

$$D = 1.25 \times 10^{-3} e^{27.8\theta} \text{ cm}^2/\text{min.}$$

The infiltration rate from the cylindrical source for both soils was found to be almost constant. Initially, the infiltration rate varied slightly, but it quickly tended toward a constant value.

The computer output contains a few terms which are explained below in Table 8.2. Later in Tables 8.3 to 8.6 the computer output for the two soils is presented. The values of time, radius and θ obtained from the solution of the problem without gravity are presented in Tables 8.3 and 8.5 for Yolo Light Clay and Pachappa Loam respectively. The gravity correction is applied to these values of θ and the result is given at angles -90°, -45°, 0°, +45° and +90° in Tables 8.4 and 8.6 corresponding to the values of radii in Tables 8.3 and 8.5.

---

[3] Ohmstede, William D., Numerical Solution of Nonsteady Soil Moisture Transfer, I., Signal Corps Meteorological Program, DA Task 3A99-27-005-08 (1957).

TABLE 8.2

Term in Output	Actual Term
DY	$y'(0)$
T	Time
R	Radius
THETA	θ
IVOL	Total volume of infiltration up to certain time. (For unit thickness of medium.)
IRATE	Rate of infiltration at any time.
IVEL	Velocity of infiltration at any time at the surface of cylindrical source.

TABLE 8.3

Solution Without Gravity for r_o = 1 cm

Yolo Light Clay

T	R	THETA
5.000000000000E+04	1.000000000000E+00	3.750000000000E-01
5.000000000000E+04	3.162277660170E+00	3.605068549480E-01
5.000000000000E+04	4.743416490250E+00	3.474076438840E-01
5.000000000000E+04	6.324555320300E+00	3.333275504530E-01
5.000000000000E+04	7.905694150400E+00	3.160479800470E-01
5.000000000000E+04	9.486832980500E+00	2.908519248740E-01
5.000000000000E+04	1.106797181060E+01	2.386552458770E-01
5.000000000000E+04	1.264911064060E+01	1.800857344700E-01
5.000000000000E+04	1.423024947090E+01	1.800002499480E-01
5.000000000000E+04	1.581138830080E+01	1.800002484350E-01
1.000000000000E+05	1.000000000000E+00	3.750000000000E-01
1.000000000000E+05	4.472135955000E+00	3.605068549480E-01
1.000000000000E+05	6.708203932500E+00	3.474076438840E-01
1.000000000000E+05	8.944271910000E+00	3.333275504530E-01
1.000000000000E+05	1.118033988750E+01	3.160479800470E-01
1.000000000000E+05	1.341640786500E+01	2.908519248740E-01
1.000000000000E+05	1.565247584250E+01	2.386552458770E-01
1.000000000000E+05	1.788854382000E+01	1.800857344700E-01
1.000000000000E+05	2.012461179750E+01	1.800002499480E-01
1.000000000000E+05	2.236067977500E+01	1.800002484350E-01
1.500000000000E+05	1.000000000000E+00	3.750000000000E-01
1.500000000000E+05	5.477225575090E+00	3.605068549480E-01
1.500000000000E+05	8.215838362600E+00	3.474076438840E-01
1.500000000000E+05	1.095445115020E+01	3.333275504530E-01
1.500000000000E+05	1.369306393770E+01	3.160479800470E-01
1.500000000000E+05	1.643167672530E+01	2.908519248740E-01
1.500000000000E+05	1.917028951280E+01	2.386552458770E-01
1.500000000000E+05	2.190890230030E+01	1.800857344700E-01
1.500000000000E+05	2.464751508790E+01	1.800002499480E-01
1.500000000000E+05	2.738612787540E+01	1.800002484350E-01
2.000000000000E+05	1.000000000000E+00	3.750000000000E-01
2.000000000000E+05	6.324555320400E+00	3.605068549480E-01
2.000000000000E+05	9.486832980500E+00	3.474076438840E-01
2.000000000000E+05	1.264911064080E+01	3.333275504530E-01
2.000000000000E+05	1.581138830090E+01	3.160479800470E-01
2.000000000000E+05	1.897366596120E+01	2.908519248740E-01
2.000000000000E+05	2.213594362140E+01	2.386552458770E-01
2.000000000000E+05	2.529822128160E+01	1.800857344700E-01
2.000000000000E+05	2.846049894170E+01	1.800002499480E-01
2.000000000000E+05	3.162277660190E+01	1.800002484350E-01
2.500000000000E+05	1.000000000000E+00	3.750000000000E-01
2.500000000000E+05	7.071067811800E+00	3.605068549480E-01
2.500000000000E+05	1.060660171770E+01	3.474076438840E-01
2.500000000000E+05	1.414213562370E+01	3.333275504530E-01
2.500000000000E+05	1.767766952950E+01	3.160479800470E-01
2.500000000000E+05	2.121320343550E+01	2.908519248740E-01
2.500000000000E+05	2.474873734140E+01	2.386552458770E-01
2.500000000000E+05	2.828427124740E+01	1.800857344700E-01

Table 8.3 (continued)

2.500000000000D+05	3.181980515340D+01	1.800002499480D-01
2.500000000000D+05	3.535533905930D+01	1.800002484350D-01
3.000000000000D+05	1.000000000000D+00	3.750000000000D-01
3.000000000000D+05	7.745966692500D+00	3.605068549480D-01
3.000000000000D+05	1.161895003870D+01	3.474076438840D-01
3.000000000000D+05	1.549193338490D+01	3.333275504530D-01
3.000000000000D+05	1.936491673110D+01	3.160479800470D-01
3.000000000000D+05	2.323790007740D+01	2.908519248740D-01
3.000000000000D+05	2.711088342360D+01	2.386552458770D-01
3.000000000000D+05	3.098386676980D+01	1.800857344700D-01
3.000000000000D+05	3.485685011600D+01	1.800002499480D-01
3.000000000000D+05	3.872983346230D+01	1.800002484350D-01
3.500000000000D+05	1.000000000000D+00	3.750000000000D-01
3.500000000000D+05	8.366600265400D+00	3.605068549480D-01
3.500000000000D+05	1.254990039800D+01	3.474076438840D-01
3.500000000000D+05	1.673320053070D+01	3.333275504530D-01
3.500000000000D+05	2.091650066340D+01	3.160479800470D-01
3.500000000000D+05	2.509980079620D+01	2.908519248740D-01
3.500000000000D+05	2.928310092870D+01	2.386552458770D-01
3.500000000000D+05	3.346640106150D+01	1.800857344700D-01
3.500000000000D+05	3.764970119420D+01	1.800002499480D-01
3.500000000000D+05	4.183300132670D+01	1.800002484350D-01

TABLE 8.4

Final Result at 45° Intervals, Moisture Content with Gravity

-90°	-45°	0°	+45°	+90°
T = 5.000000000000E+04				
3.750000000000E-01	3.750000000000E-01	3.750000000000E-01	3.750000000000E-01	3.750000000000E-01
3.606336666633E-01	3.605986178831E-01	3.605068549488E-01	3.604171990658E-01	3.603800432328E-01
3.475771247898E-01	3.475274666863E-01	3.474076438848E-01	3.472878208848E-01	3.472381629788E-01
3.335482770908E-01	3.334836041858E-01	3.333275504538E-01	3.331714967198E-01	3.331068238168E-01
3.163740124678E-01	3.162788489698E-01	3.160479800478E-01	3.158174751248E-01	3.157219476268E-01
2.913903606048E-01	2.912325989378E-01	2.908519248748E-01	2.904712508118E-01	2.903134891448E-01
2.897522498778E-01	2.394308277048E-01	2.386552458778E-01	2.378796640528E-01	2.375582418808E-01
1.803778753468E-01	1.802922780688E-01	1.800857344708E-01	1.800000000008E-01	1.800000000008E-01
1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01

AT TIME= 5.000000000008E+04 IVOL= 7.124784441208E+01 IRATE= 1.424956888248E-03 IVEL= 2.267889363258E-04

T = 1.000000000008E+05				
3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01
3.607171783308E-01	3.606556903538E-01	3.605068549488E-01	3.603573572178E-01	3.602952065098E-01
3.476963305008E-01	3.476118768298E-01	3.474076438848E-01	3.472027749688E-01	3.471176849438E-01
3.337033379698E-01	3.335934429658E-01	3.333275504538E-01	3.330606413418E-01	3.329497285278E-01
3.166025829478E-01	3.164404922438E-01	3.160479800478E-01	3.156534992938E-01	3.154894388628E-01
2.917680720298E-01	2.915002682048E-01	2.908519248748E-01	2.902005736698E-01	2.899297601198E-01
2.405195970478E-01	2.399750761158E-01	2.386552458778E-01	2.373270941248E-01	2.367742477228E-01
1.806010191278E-01	1.804465392718E-01	1.800857344708E-01	1.800000000008E-01	1.800000000008E-01
1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01

AT TIME= 1.000000000008E+05 IVOL= 1.432943911588E+02 IRATE= 1.440930934918E-03 IVEL= 2.293312848598E-04

T = 1.500000000008E+05				
3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01
3.607833618718E-01	3.607026637178E-01	3.605068549488E-01	3.603094989388E-01	3.602272525748E-01
3.477932714098E-01	3.476806018208E-01	3.474076438848E-01	3.471331377668E-01	3.470189190578E-01
3.338293645018E-01	3.336828343798E-01	3.333275504538E-01	3.329698322258E-01	3.328208663368E-01
3.167880843748E-01	3.165722063428E-01	3.160479800478E-01	3.155190413548E-01	3.152984481778E-01
2.920747310148E-01	2.917179495078E-01	2.908519248748E-01	2.899787000198E-01	2.896147142068E-01
2.411413657178E-01	2.404170639738E-01	2.386552458778E-01	2.368735088318E-01	2.361292816498E-01
1.807930648478E-01	1.805773356818E-01	1.800857344708E-01	1.800000000008E-01	1.800000000008E-01
1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01	1.800000000008E-01

AT TIME= 1.500000000008E+05 IVOL= 2.153550451328E+02 IRATE= 1.441213079478E-03 IVEL= 2.293761895588E-04

T = 2.000000000008E+05				
3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01	3.750000000008E-01
3.608396553558E-01	3.607426677968E-01	3.605068549488E-01	3.602684930378E-01	3.601689547778E-01
3.478769435908E-01	3.477399783728E-01	3.474076438848E-01	3.470726866008E-01	3.469330969838E-01

Table 8.4 (continued)

3.33938099656@-01	3.33760045219@-01	3.33327550453@-01	3.32890978709@-01	3.32708844796@-01
3.16947943073@-01	3.16685877087@-01	3.16047980047@-01	3.15402190739@-01	3.15132227943@-01
2.92339075496@-01	2.91905852414@-01	2.90851924874@-01	2.89785938794@-01	2.89340651038@-01
2.41675488503@-01	2.40798156554@-01	2.38655245877@-01	2.36478977375@-01	2.35567281400@-01
1.80966496628@-01	1.80694086791@-01	1.80085734470@-01	1.80000000000@-01	1.80000000000@-01
1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01

AT TIME= 2.00000000000@+05 IVOL= 2.87427073374@+02 IRATE= 1.44144056484@-03 IVEL= 2.29412394980@-04

T = 2.50000000000@+05

3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01
3.60889370133@-01	3.60778034890@-01	3.60506854948@-01	3.60232050302@-01	3.60117087941@-01
3.47951562660@-01	3.47792976054@-01	3.47407643884@-01	3.47018504408@-01	3.46856108132@-01
3.34035042045@-01	3.33828946874@-01	3.33327550453@-01	3.32820286189@-01	3.32608319486@-01
3.17090316486@-01	3.16787241031@-01	3.16047980047@-01	3.15297360141@-01	3.14982919283@-01
2.92574560380@-01	2.92073442017@-01	2.90851924874@-01	2.89613053037@-01	2.89094572289@-01
2.42152542848@-01	2.41137720496@-01	2.38655245877@-01	2.36124764208@-01	2.35061932367@-01
1.81127218927@-01	1.80801240838@-01	1.80085734470@-01	1.80000000000@-01	1.80000000000@-01
1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01

AT TIME= 2.50000000000@+05 IVOL= 3.59508829282@+02 IRATE= 1.44163511816@-03 IVEL= 2.29443359104@-04

T = 3.00000000000@+05

3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01
3.60934312180@-01	3.60810037323@-01	3.60506854948@-01	3.60198920604@-01	3.60069890492@-01
3.48019500151@-01	3.47841265523@-01	3.47407643884@-01	3.46968949742@-01	3.46785639392@-01
3.34123283514@-01	3.33891716829@-01	3.33327550453@-01	3.32755618793@-01	3.32516281784@-01
3.17219706758@-01	3.16879524058@-01	3.16047980047@-01	3.15201404241@-01	3.14846097304@-01
2.92788750920@-01	2.92226041308@-01	2.90851924874@-01	2.89454842926@-01	2.88869159238@-01
2.42585029204@-01	2.41446644772@-01	2.38655245877@-01	2.35800321058@-01	2.34598415901@-01
1.81278506448@-01	1.80901311038@-01	1.80085734470@-01	1.80000000000@-01	1.80000000000@-01
1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01

AT TIME= 3.00000000000@+05 IVOL= 4.31599184686@+02 IRATE= 1.44180710807@-03 IVEL= 2.29470732149@-04

T = 3.50000000000@+05

3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01	3.75000000000@-01
3.60975591779@-01	3.60839457424@-01	3.60506854948@-01	3.60168334705@-01	3.60026278335@-01
3.48082245488@-01	3.47885896073@-01	3.47407643884@-01	3.46922991496@-01	3.46720237730@-01
3.34204765256@-01	3.33949722399@-01	3.33327550453@-01	3.32695634567@-01	3.32430841221@-01
3.17339244574@-01	3.16964751626@-01	3.16047980047@-01	3.15112366689@-01	3.14718981311@-01
2.92986396774@-01	2.92366992676@-01	2.90851924874@-01	2.89308041157@-01	2.88659812532@-01
2.42983676548@-01	2.41731765583@-01	2.38655245877@-01	2.35499019416@-01	2.34167418673@-01
1.81422685384@-01	1.80995879992@-01	1.80085734470@-01	1.80000000000@-01	1.80000000000@-01
1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01	1.80000000000@-01

AT TIME= 3.50000000000@+05 IVOL= 5.03697396126@+02 IRATE= 1.44196242880@-03 IVEL= 2.29495452211@-04

TABLE 8.5

Solution without Gravity for r_o = 1 cm.

Pachappa Loam

T	R	THETA
5.000000000000E+00	1.000000000000E+00	3.300000000000E-01
5.000000000000E+00	2.795084971880E+00	3.183821873550E-01
5.000000000000E+00	3.354101966250E+00	3.052494734050E-01
5.000000000000E+00	3.913118960630E+00	2.891437592490E-01
5.000000000000E+00	4.472135955000E+00	2.668922211950E-01
5.000000000000E+00	5.031152949380E+00	2.278103165490E-01
5.000000000000E+00	5.590169943800E+00	1.051631998360E-01
5.000000000000E+00	6.149186938100E+00	9.999090728000E-02
5.000000000000E+00	6.708203932500E+00	9.999090309000E-02
1.000000000000E+01	1.000000000000E+00	3.300000000000E-01
1.000000000000E+01	3.952847075240E+00	3.183821873550E-01
1.000000000000E+01	4.743416490300E+00	3.052494734050E-01
1.000000000000E+01	5.533985905300E+00	2.891437592490E-01
1.000000000000E+01	6.324555320400E+00	2.668922211950E-01
1.000000000000E+01	7.115124735400E+00	2.278103165490E-01
1.000000000000E+01	7.905694150500E+00	1.051631998360E-01
1.000000000000E+01	8.696263565600E+00	9.999090728000E-02
1.000000000000E+01	9.486832980500E+00	9.999090309000E-02
1.500000000000E+01	1.000000000000E+00	3.300000000000E-01
1.500000000000E+01	4.841229182780E+00	3.183821873550E-01
1.500000000000E+01	5.809475019300E+00	3.052494734050E-01
1.500000000000E+01	6.777720855900E+00	2.891437592490E-01
1.500000000000E+01	7.745966692500E+00	2.668922211950E-01
1.500000000000E+01	8.714212529000E+00	2.278103165490E-01
1.500000000000E+01	9.682458365600E+00	1.051631998360E-01
1.500000000000E+01	1.065070420210E+01	9.999090728000E-02
1.500000000000E+01	1.161895003870E+01	9.999090309000E-02
2.000000000000E+01	1.000000000000E+00	3.300000000000E-01
2.000000000000E+01	5.590169943800E+00	3.183821873550E-01
2.000000000000E+01	6.708203932500E+00	3.052494734050E-01
2.000000000000E+01	7.826237921300E+00	2.891437592490E-01
2.000000000000E+01	8.944271910000E+00	2.668922211950E-01
2.000000000000E+01	1.006230589880E+01	2.278103165490E-01
2.000000000000E+01	1.118033988750E+01	1.051631998360E-01
2.000000000000E+01	1.229837387630E+01	9.999090728000E-02
2.000000000000E+01	1.341640786500E+01	9.999090309000E-02
2.500000000000E+01	1.000000000000E+00	3.300000000000E-01
2.500000000000E+01	6.250000000000E+00	3.183821873550E-01
2.500000000000E+01	7.500000000000E+00	3.052494734050E-01
2.500000000000E+01	8.750000000000E+00	2.891437592490E-01
2.500000000000E+01	1.000000000000E+01	2.668922211950E-01
2.500000000000E+01	1.125000000000E+01	2.278103165490E-01
2.500000000000E+01	1.250000000000E+01	1.051631998360E-01
2.500000000000E+01	1.375000000000E+01	9.999090728000E-02
2.500000000000E+01	1.500000000000E+01	9.999090309000E-02
3.000000000000E+01	1.000000000000E+00	3.300000000000E-01
3.000000000000E+01	6.846531968800E+00	3.183821873550E-01

Table 8.5 (continued)

3,000000000000@+01	8,215838362600@+00	3,052494734050@-01
3,000000000000@+01	9,585144756400@+00	2,891437592490@-01
3,000000000000@+01	1,095445115020@+01	2,668922211950@-01
3,000000000000@+01	1,232375754380@+01	2,278103165490@-01
3,000000000000@+01	1,369306393750@+01	1,051631998360@-01
3,000000000000@+01	1,506237033140@+01	9,999090728000@-02
3,000000000000@+01	1,643167672520@+01	9,999090309000@-02
3,500000000000@+01	1,000000000000@+00	3,300000000000@-01
3,500000000000@+01	7,395099728900@+00	3,183821873550@-01
3,500000000000@+01	8,874119674700@+00	3,052494734050@-01
3,500000000000@+01	1,035313962040@+01	2,891437592490@-01
3,500000000000@+01	1,183215956630@+01	2,668922211950@-01
3,500000000000@+01	1,331117951190@+01	2,278103165490@-01
3,500000000000@+01	1,479019945780@+01	1,051631998360@-01
3,500000000000@+01	1,626921940350@+01	9,999090728000@-02
3,500000000000@+01	1,774823934920@+01	9,999090309000@-02

TABLE 8.6

Final Result at 45° Intervals, Moisture Content with Gravity

-90°	-45°	0°	+45°	+90°
		T = 5.000000000000E+00		
3.300000000000E-01	3.300000000000E-01	3.300000000000E-01	3.300000000000E-01	3.300000000000E-01
3.184660011633E-01	3.184414440499E-01	3.183821873559E-01	3.183229306619E-01	3.182983730799E-01
3.054221344369E-01	3.053715447539E-01	3.052494734059E-01	3.051274020569E-01	3.050768123729E-01
2.893944266999E-01	2.893209811379E-01	2.891437592499E-01	2.889665373639E-01	2.888930917999E-01
2.673120998019E-01	2.671890753699E-01	2.668922211959E-01	2.665953670219E-01	2.664723425499E-01
2.286076096639E-01	2.283740027799E-01	2.278103165499E-01	2.272466303179E-01	2.270130234339E-01
1.068344601119E-01	1.063447808509E-01	1.051631998369E-01	1.039816188209E-01	1.034919395599E-01
1.000020650879E-01	1.000012077399E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01
1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01

AT TIME= 5.000000000009E+00 IVOL= 2.236719866479E+01 IRATE= 4.473439732949E+00 IVEL= 7.119700582709E-01

		T = 1.000000000009E+01		
3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01
3.185227276709E-01	3.184816346379E-01	3.183821873559E-01	3.182823282939E-01	3.182406232309E-01
3.055433235839E-01	3.054574118459E-01	3.052494734059E-01	3.050406333199E-01	3.049538194329E-01
2.895703427809E-01	2.894562977269E-01	2.891437592499E-01	2.888405557529E-01	2.887145088729E-01
2.676061869039E-01	2.673975775089E-01	2.668922211959E-01	2.663840543149E-01	2.661726326719E-01
2.291666511349E-01	2.287702254039E-01	2.278103165499E-01	2.268456828249E-01	2.264445295249E-01
1.080358920919E-01	1.071901331719E-01	1.051631998369E-01	1.031560301269E-01	1.023300416079E-01
1.000045498279E-01	1.000028751029E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01
1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01

AT TIME= 1.000000000009E+01 IVOL= 4.572079942119E+01 IRATE= 4.670720151289E+00 IVEL= 7.433682125609E-01

		T = 1.500000000009E+01		
3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01
3.185681925329E-01	3.185138950309E-01	3.183821873559E-01	3.182495010819E-01	3.181942243839E-01
3.056417500059E-01	3.055272592199E-01	3.052494734059E-01	3.049695198249E-01	3.044525599389E-01
2.897132164489E-01	2.895470231159E-01	2.891437592499E-01	2.887373043369E-01	2.885679180409E-01
2.678447008679E-01	2.675670131079E-01	2.668922211959E-01	2.662107012289E-01	2.659267814279E-01
2.296203908269E-01	2.290923803949E-01	2.278103165499E-01	2.265169423169E-01	2.259776154889E-01
1.090278502289E-01	1.078859201879E-01	1.051631998369E-01	1.024877776969E-01	1.013931363949E-01
1.000057883029E-01	1.000034439409E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01
1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01	1.000000000009E-01

AT TIME= 1.500000000009E+01 IVOL= 6.907493570009E+01 IRATE= 4.670827255749E+00 IVEL= 7.433852587609E-01

		T = 2.000000000009E+01		
3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01	3.300000000009E-01
3.186071141009E-01	3.185415464009E-01	3.183821873559E-01	3.182211970729E-01	3.181539971249E-01
3.057266195239E-01	3.055875627119E-01	3.052494734059E-01	3.049077426719E-01	3.047650421149E-01

Table 8.6 (continued)

2.898364144338-01	2.896345638458-01	2.891437592498-01	2.886476102878-01	2.884404120978-01
2.680501316698-01	2.677131817978-01	2.668922211958-01	2.660599924758-01	2.657117677858-01
2.300114281958-01	2.297704161878-01	2.278103165498-01	2.262312747678-01	2.255713114258-01
1.098944853778-01	1.094923531908-01	1.051631998368-01	1.019132365738-01	1.005902294088-01
1.000089046918-01	1.000056970928-01	1.000000000008-01	1.000000000008-01	1.000000000008-01
1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01

AT TIME= 2.000000000008+01 IVOL= 9.242968024108+01 IRATE= 4.670948908298+00 IVEL= 7.434046203708-01

T = 2.500000000008+01

3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01
3.186416352908-01	3.185660976308-01	3.183821873558-01	3.181959372038-01	3.181180582088-01
3.058022430438-01	3.056413557638-01	3.052494734058-01	3.048523383688-01	3.046861949328-01
2.899461947668-01	2.897126555568-01	2.891437592498-01	2.885671710198-01	2.883259351518-01
2.682330056208-01	2.678043481768-01	2.668922211958-01	2.659247429278-01	2.655189916498-01
2.303597059238-01	2.296183581118-01	2.278103165498-01	2.259750131688-01	2.252063916728-01
1.106753607328-01	1.090377114778-01	1.051631998368-01	1.014026241968-01	1.000000000008-01
1.000109438078-01	1.000069737168-01	1.000000000008-01	1.000000000008-01	1.000000000008-01
1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01

AT TIME= 2.500000000008+01 IVOL= 1.157871577078+02 IRATE= 4.671495493128+00 IVEL= 7.434916120508-01

T = 3.000000000008+01

3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01
3.186729396158-01	3.185883823808-01	3.183821873558-01	3.181729037688-01	3.180852557718-01
3.058710446628-01	3.056903444468-01	3.052494734058-01	3.048016388998-01	3.046139704048-01
2.900460759918-01	2.897837747468-01	2.891437592498-01	2.884935645858-01	2.882210777728-01
2.683992400138-01	2.679620730578-01	2.668922211958-01	2.658009075698-01	2.653422657958-01
2.306764362538-01	2.298440933858-01	2.278103165498-01	2.257404637448-01	2.248720325358-01
1.113927813458-01	1.095379275268-01	1.051631998368-01	1.009391957608-01	1.000000000008-01
1.000129335568-01	1.000081958988-01	1.000000000008-01	1.000000000008-01	1.000000000008-01
1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01

AT TIME= 3.000000000008+01 IVOL= 1.391569225548+02 IRATE= 4.673952969538+00 IVEL= 7.438827315708-01

T = 3.500000000008+01

3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01	3.300000000008-01
3.187017620078-01	3.186089178348-01	3.183821873558-01	3.181515891968-01	3.180548747558-01
3.059345463558-01	3.057356000818-01	3.052494734058-01	3.047545962768-01	3.045468932298-01
2.901382670488-01	2.898494764178-01	2.891437592498-01	2.884252692308-01	2.881236975218-01
2.685525492318-01	2.680715675358-01	2.668922211958-01	2.656859446368-01	2.651780160658-01
2.309686580858-01	2.300525743338-01	2.278103165498-01	2.255227920968-01	2.245614364778-01
1.120607785988-01	1.100030187028-01	1.051631998368-01	1.005124383018-01	1.000000000008-01
1.000148907708-01	1.000093775028-01	1.000000000008-01	1.000000000008-01	1.000000000008-01
1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01	1.000000000008-01

AT TIME= 3.500000000008+01 IVOL= 1.625305475198+02 IRATE= 4.674724992928+00 IVEL= 7.440056029108-01

For these soils, the moisture profiles with or without gravity are very similar, hence only one plot is presented for each soil (Figure 8.1 - Yolo Light Clay, Figure 8.3 - Pachappa Loam). The moisture profiles on the vertical line for the "with gravity" and "without gravity" conditions will become more divergent at larger times. Infiltration rate versus time is plotted for these soils in Figures 8.2 and 8.4.

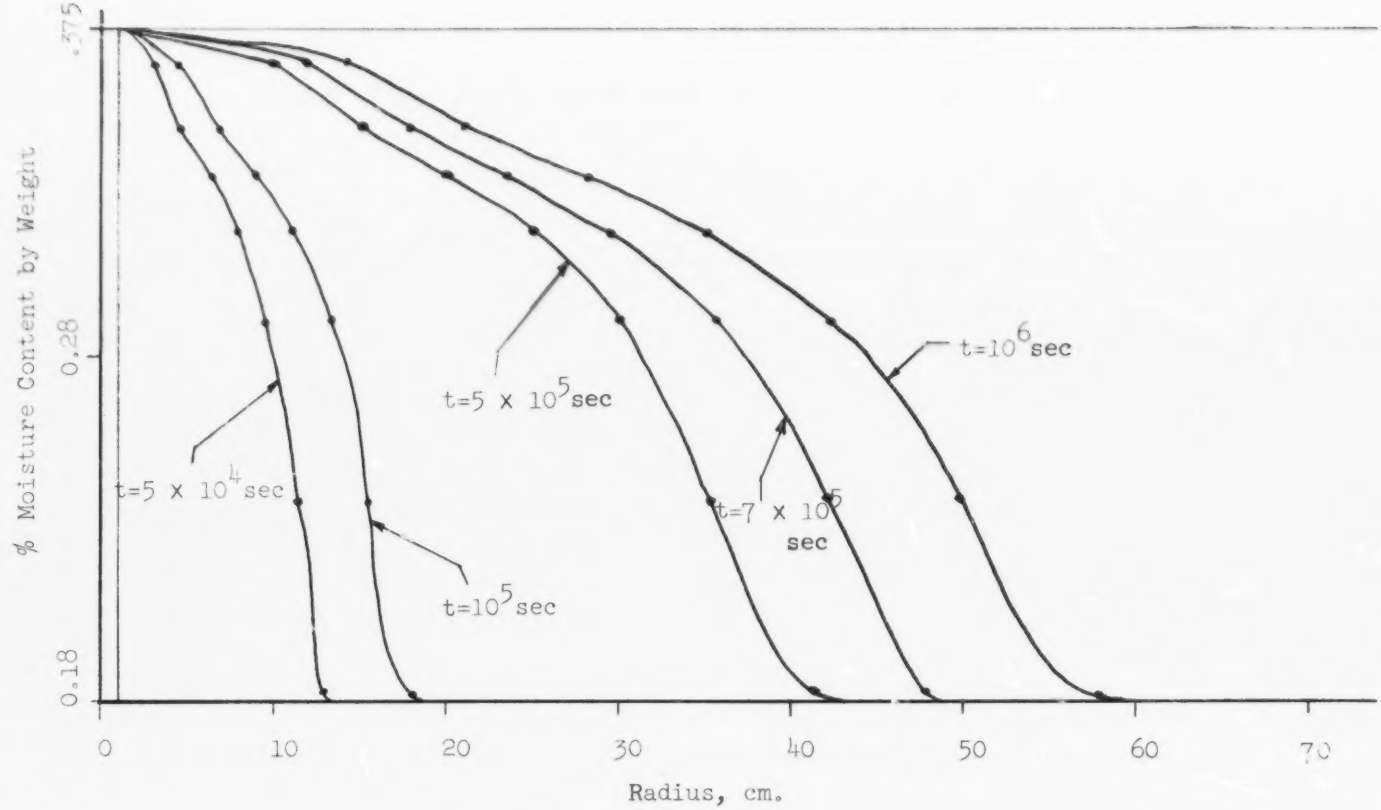


Fig. 8.1: Moisture Profile at Different Times for Yolo Light Clay.

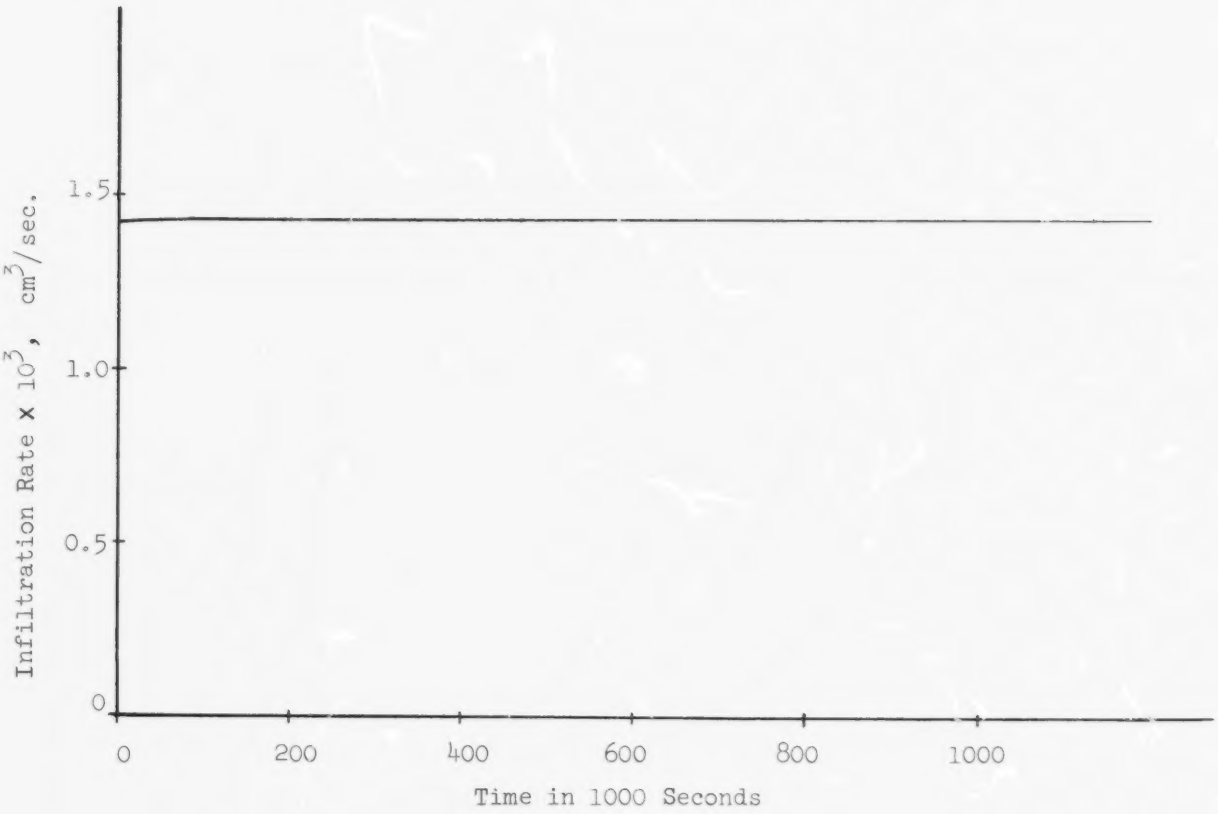


Fig. 8.2: Infiltration Curve for Yolo Light Clay.

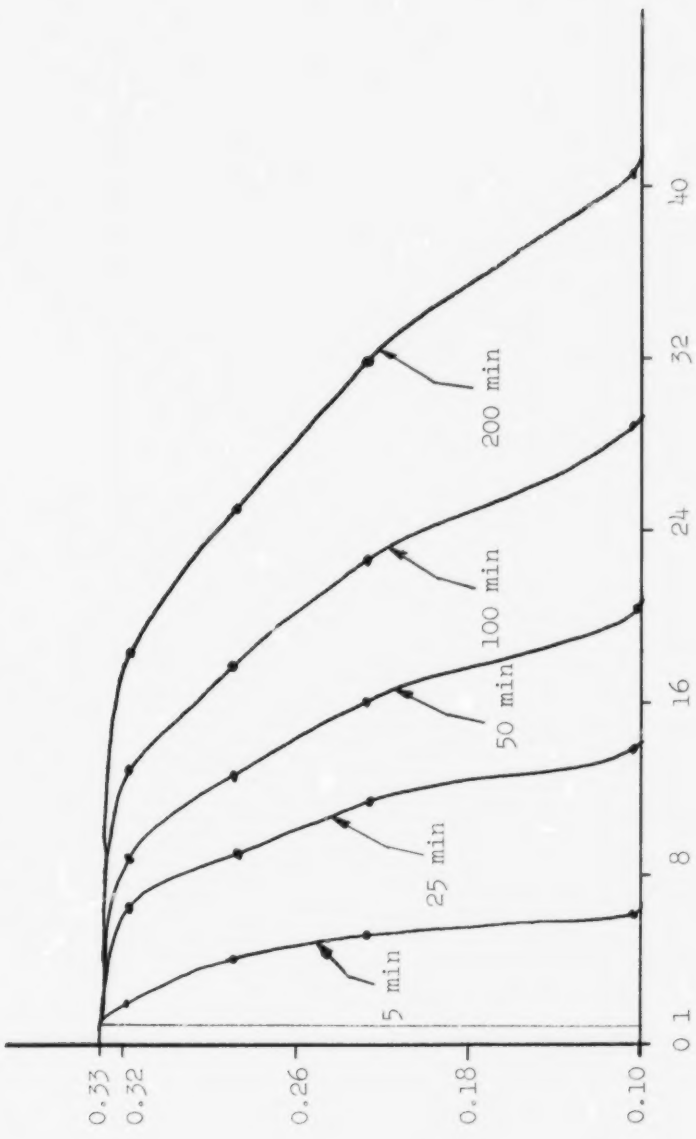


Fig. 8.3: Moisture Profiles for Pachappa Loam

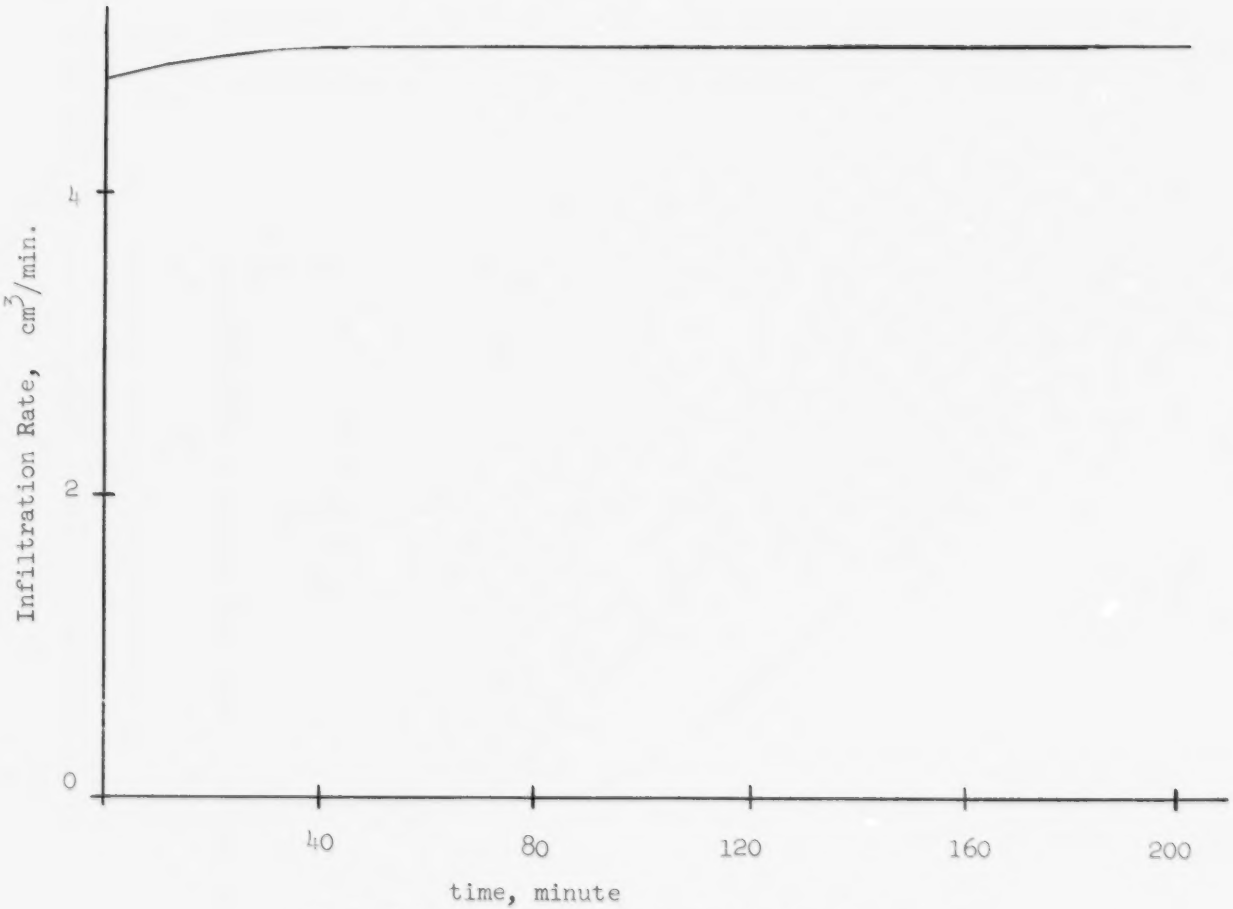


Fig. 8.4: Infiltration Rate for Pachappa Loam.

The infiltration rate increases with increase in the size of the cylindrical source. Infiltration rates using the computer program were computed for Pachappa Loam for sources of radii 1, 2 and 5 centimeters and are presented in Table 8.7. The values of infiltration rates in Table 8.7 are taken at 35 minutes after starting the flow to avoid initial variations in the rate.

TABLE 8.7

Variation of Infiltration Rate with Radius of the Source

r_o (cm)	Infiltration Rate (cm^3/min.)
1	4.67
2	9.12
5	31.63

CHAPTER 9

CONCLUSION

The main purpose of this study was to solve the general two-dimensional diffusion equation for the problem of unsteady and unsaturated flow outward from a cylindrical source. The problem has been solved in such a way that solutions, with and without gravity, are obtained at the same time. An attempt was made to carry the solution of the equation analytically as far as possible. Due to nonlinearity of the equation, resort had to be made to numerical techniques, but the selection of these techniques was made with due regard to maximum accuracy and efficient computer programming. Two soils have been selected for examples and the results are consistent and satisfactory.

A computer model has been set up in extended Algol language for the Burroughs B 5500 machine, which can be applied to similar problems in other soils having an exponential diffusivity relationship. To use this model, the relation of the hydraulic conductivity and the diffusivity to the saturation of a soil must be known. The application of this model may be made in problems of diffusion of water from porous underground pipes, disposal of nuclear wastes with liquids in deep wells and others. The model can be used for both draining and wetting of soils, the only difference will be in the selection of values to be fed into the computer. The stability of numerical techniques, applied in this study, is extremely good. If the user of the model is careful with the size of the steps in the numerical integration, there should be no problem. A few trials with this model will be sufficient to acquaint one with its working.

Two important observations of this study are:

(1) The wetting front, moving away from the cylindrical source in two-dimensional problem, moves slower than the wetting front under similar circumstances in the one-dimensional problem. This seems reasonable because for the problem of the two-dimensional cylindrical source, the wetting front has to spread out over a larger and larger area as it moves away from the source and this slows down the progress. In the case of the one-dimensional problem, the area is constant and the wetting front does not have to spread out laterally.

(2) Infiltration rate from the cylindrical source in the two-dimensional problem remains almost constant with time, provided the soil is homogeneous and isotropic and the soil properties do not change with time as assumed in the solution. This result can be justified, when it is approximated that the rate of infiltration is proportional to the rate of increase of area behind the wetting front with time. It is observed from the solution that the distance of the wetting front from the source is approximately proportional to the square-root of time, and the area behind the wetting front is proportional to the square of the distance of the wetting front from the source; hence the accumulated volume of infiltration is nearly proportional to time and thus the rate of infiltration is approximately constant.

In this study the equation has been solved and boundary conditions satisfied. Experimentally determined and published values of the hydraulic conductivity and diffusivity have been used for the soils. The final solution has not been verified experimentally. An experimental

program would be a logical next step in the research on this problem.

In this new field of unsteady and unsaturated flow in soils, vast areas of future research exist. Some further research possibilities originating from this study are summarized below:

(a) The scope of the two-dimensional solution for a cylindrical source may be enlarged to include sources of other shapes such as an elliptical source.

(b) In connection with two-dimensional studies, the writer did some work on the general three-dimensional equation which is presented in Appendix 2. Further investigation may be made to solve the three-dimensional problem.

(c) The writer initially tried to solve this problem by an implicit difference method and encountered difficulties with the stability of the solution. However, in the opinion of the writer, this technique has a bright prospect and the appropriate criterion of stability of the finite difference technique in this problem may be found by further investigation.

(d) In this study, curves have been fitted to relate the hydraulic conductivity and the diffusivity to the saturation of the soils. Though the curves fit closely to the experimental points, the use of experimental data directly in the solution of the problem is possible. This can be done by using the two-dimensional equation in polar coordinates and some of the writer's transformations, and attempting to solve the equation numerically without using the empirical relations for the above two parameters.

(e) The two-dimensional problem has been first solved without gravity, and then an approximate correction for gravity has been applied. In this connection, some terms have been neglected, and the solution thus obtained seems satisfactory. In the examples chosen, the effect of gravity is extremely small and the further addition of neglected terms will only change the result at higher decimal places, but the complexity of the solution will be proportionately much higher. In larger grained soils where the influence of gravity is significant, the neglected terms of the equation should be considered. Further work along these lines would be useful.

(f) If proper boundary conditions can be defined for the diffusion of water from a semicircular canal, then solution of the two-dimensional equation in polar coordinates may be attempted. Evaporation will take place from the ground surface on both sides of the canal by unsaturated mechanism, and the boundary conditions at the ground surface become involved because of the complex nature of the evaporation phenomenon. This is a very interesting area for future investigation. Initially at least, some idealized boundary conditions may be assumed at the ground surface.

Many other research possibilities exist in this area; it is hoped that this study will be of some significance in pointing out the direction in which further investigations are warranted.

APPENDIX 1

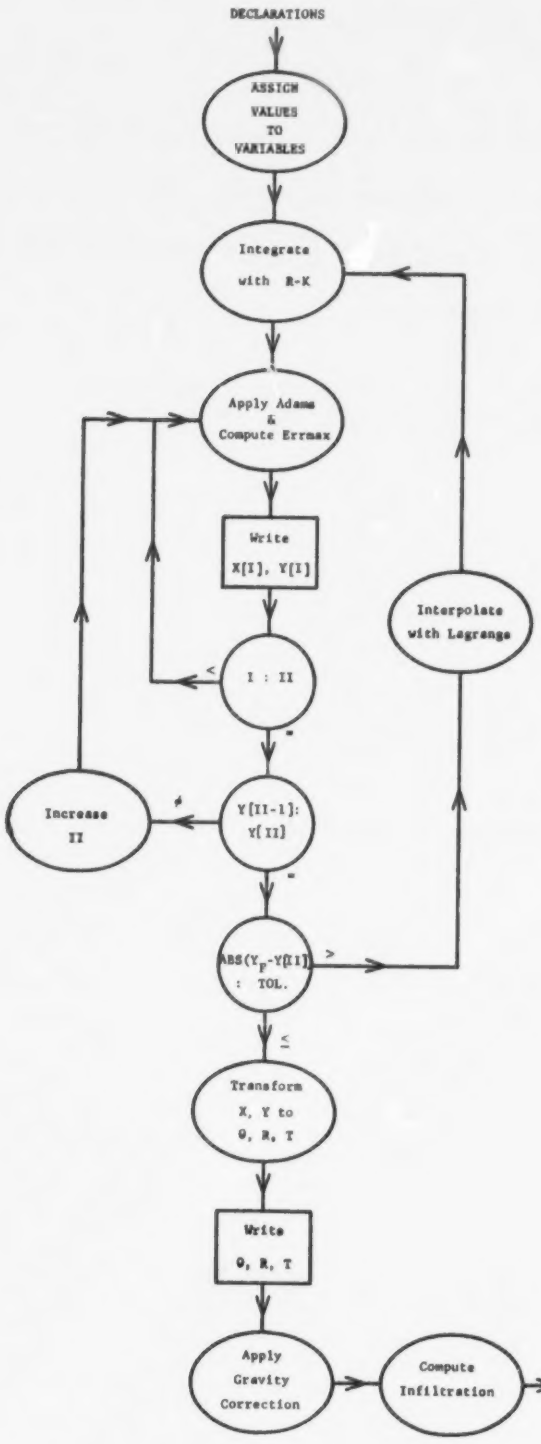
A. ALGOL PROGRAM

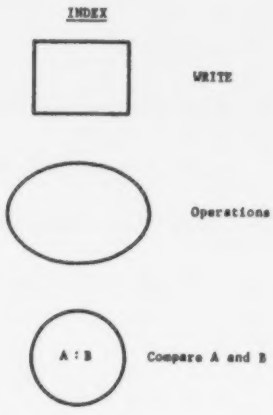
```
BEGIN COMMENT RAMESHWAR SINGH,G259,BOX 149,                                    001
UNSTEADY AND UNSATURATED FLOW IN SOILS IN TWO DIMENSIONS,                      002
ALGOL PROGRAM FOR EXAMPLE INTO PACHAPPA LOAM;                                  003
FORMAT SL(/X10,"DY =",F20.11/),                                                004
       HEAD(/X10,"T",X19,"R",X18,"THETA"),                                     005
       HEAD1(3E20.11),                                                         006
       HEAD2(/3E20.11),                                                        007
       HEADING(/X9,"X",X20,"Y"/),                                              008
       VALUES(2E20.11),                                                        009
       ER(//X6,"MAX ERROR =",E20.11/);                                         010
REAL MAX,MIN,TEST,THETA,RETAIN,M0,M1,M2,M3,M,ERRMAX,ANS,AL,BET,TMX,TMN;        011
INTEGER I,J,II,K,P;                                                            012
LABEL BACK,AHEAD,INTERPOL,LAST,RETURN,REPEAT;                                  013
BOOLEAN CHK;                                                                   014
REAL ARRAY X[0:800],Y,DY[0:800],YKEAP,DYKEAP[0:10],G[0:2];                     015
COMMENT PROCEDURE FOR INTERPOLATION;                                           016
PROCEDURE LAGRANGE(N,U,X,Y,ANS);                                               017
ARRAY X,Y[1];                                                                  018
INTEGER N; REAL U,ANS;                                                         019
BEGIN INTEGER I,J; REAL L;                                                     020
ANS←0;                                                                         021
FOR J←1 STEP 1 UNTIL N+1 DO BEGIN L←1;                                         022
FOR I←1 STEP 1 UNTIL N+1 DO                                                    023
BEGIN IF I≠J THEN L←L×(U-X[I])/(X[J]-X[I]);END;                                024
ANS←ANS+L×Y[J];END;END LAGRANGE;                                               025
REAL PROCEDURE M(X,Y,DY);                                                      026
VALUE X,Y,DY;                                                                  027
REAL X,Y,DY;                                                                   028
M←-DY×(1/X+X/Y);                                                               029
COMMENT ASSIGN VALUES TO CONSTANTS;                                            030
    AL←1.25@-3;                                                                031
    BET←27.8;                                                                  032
    TMX←0.33;                                                                  033
    TMN←0.10;                                                                  034
    DY[0]←-599.78;                                                             035
    G[1]←-600;                                                                 036
    G[2]←-602;                                                                 037
    II←400;                                                                    038
    H←0.10;                                                                    039
    X[0]←1/SQRT(2×AL×1.0);                                                     040
J←0; MAX←EXP(BET×TMX);MIN←EXP(BET×TMN);                                        041
Y[0]←MAX;                                                                      042
FOR I←1 STEP 1 UNTIL II DO X[I]←X[0]+I×H;                                      043
ERRMAX←I←K←0;                                                                  044
RETURN:WRITE([PAGE]);                                                          045
       WRITE(SL,DY[I]);                                                        046
       WRITE(HEADING);                                                         047
BACK:  IF I/5 = ENTIER(I/5) THEN                                               048
       WRITE(VALUES,X[I],Y[I]);                                                049
       IF I=3 THEN BEGIN I←4;GO TO AHEAD;END;                                  050
COMMENT RUNGEKUTTA AND ADAMS METHOD;                                           051
    M0←M(X[I],Y[I],DY[I]);                                                     052
```

```
     M1←M(X[I]+0.5×H,Y[I]+0.5×H×DY[I],DY[I]+0.5×M0);                                  053
     M2←M(X[I]+0.5×H,Y[I]+0.5×H×DY[I]+0.25×H×M0,DY[I]+0.5×M1);                        054
     M3←M(X[I]+H,Y[I]+H×DY[I]+0.5×H×M1,DY[I]+M2);                                     055
     Y[I+1]←Y[I]+H×DY[I]+H×H×(M0+M1+M2)/6;                                            056
     DY[I+1]←DY[I]+H×(M0+2×M1+2×M2+M3)/6;                                             057
     I←I+1;                                                                           058
     GO TO BACK;                                                                      059
AHEAD:DY[I]←DY[I-1]+H×(55×M(X[I-1],Y[I-1],DY[I-1])-59×M(X[I-2],Y[I-2],DY              060
[I-2])+37×M(X[I-3],Y[I-3],DY[I-3])-9×M(X[I-4],Y[I-4],DY[I-4]))/24;                    061
Y[I]←Y[I-1]+H×(55×DY[I-1]-59×DY[I-2]+37×DY[I-3]-9×DY[I-4])/24;                       062
REPEAT:DY[I]←DY[I-1]+H×(9×M(X[I],Y[I],DY[I])+19×M(X[I-1],Y[I-1],DY[I-1])             063
-5×M(X[I-2],Y[I-2],DY[I-2])+M(X[I-3],Y[I-3],DY[I-3]))/24;                             064
TEST←Y[I];                                                                           065
Y[I]←Y[I-1]+H×(9×DY[I]+19×DY[I-1]-5×DY[I-2]+DY[I-3])/24;                             066
K←K+1;                                                                               067
IF K≤1 THEN GO TO REPEAT; K←0;                                                       068
RETAIN←ABS(TEST-Y[I])/18;                                                            069
IF RETAIN>ERRMAX THEN ERRMAX←RETAIN;                                                 070
IF I/5 = ENTIER(I/5) THEN                                                            071
WRITE(VALUES,X[I],Y[I]);                                                             072
I←I+1;                                                                               073
IF I≤II THEN GO TO AHEAD; I←I-1;                                                     074
IF Y[I]-Y[I-1]≠0 THEN BEGIN                                                          075
FOR P←1 STEP 1 UNTIL 40 DO X[II+P]←X[II]+P×H;                                        076
II←II+40;I←I+1;GO TO AHEAD;END;                                                      077
YKEAP[J]←Y[I];                                                                       078
DYKEAP[J]←DY[0];                                                                     079
WRITE(ER,ERRMAX);                                                                    080
J←J+1; ERRMAX←0;                                                                     081
IF ABS(Y[I]-MIN)<0.1 THEN GO TO LAST                                                 082
ELSE IF J≥3 THEN GO TO INTERPOL ELSE                                                 083
BEGIN I←0;DY[I]←G[J];GO TO RETURN;END;                                               084
INTERPOL:LAGRANGE(J,MIN,YKEAP,DYKEAP,ANS);                                           085
I←0;DY[I]←ANS;GO TO RETURN;                                                          086
LAST:BEGIN COMMENT CALCULATION OF RADIUS AND TIME;                                   087
     FORMAT EF(/X40,"MOISTURE CONTENT WITH GRAVITY"//),                              088
            T(/X40,"T =",E20.11/),                                                   089
            INF(/"AT TIME=",E18.11,X2,"IVOL=",E18.11,X2,"IRATE=",E18.11,             090
            X2,"IVEL=",E18.11/),                                                     091
            GR(5E20.11);                                                             092
     INTEGER K,L,M;                                                                  093
     REAL TEMP,TOTAL,TOTAL1,INFILT,IRATE,IVEL,SG;                                    094
     REAL ARRAY MOIST,D,KP[0:4,0:9],R[0:40,0:9],F[0:4];                              095
     LABEL FINAL,HARD;                                                               096
COMMENT INITIL R EQUALS ONE AND DT 5;                                                097
WRITE(<PAGE>);                                                                       098
WRITE(HEAD);                                                                         099
K←L←M←0;                                                                             100
CHK←FALSE;                                                                           101
TOTAL←TOTAL1←INFILT←0;                                                               102
FOR I←0 STEP 1 UNTIL 4 DO BEGIN                                                      103
    MOIST[I,0]←TMX;                                                                  104
    KP[I,0]←6.25@-3;                                                                 105
    D[I,0]←MAX×AL;END;                                                               106
FOR I←0 STEP 1 UNTIL 40 DO R[I,0]←1;                                                 107
```

```
FOR I←50 STEP 50 UNTIL II  DO BEGIN L←L+1; TEST←(LN(Y[I]))/BET;                108
FOR J←0 STEP 1 UNTIL 4 DO MOIST[J,L]←TEST;END;                                 109
P←8;                                                                           110
HARD:FOR I←1 STEP 1 UNTIL P DO FOR J←0 STEP 1 UNTIL 4 DO                       111
     BEGIN D[J,I]←EXP(BET×MOIST[J,I])×AL;                                      112
     KP[J,I]←6.25@-3×((MOIST[J,I]-.04)/(TMX-.04))↑6.0;END;                     113
IF CHK THEN GO TO FINAL; L←0;                                                  114
COMMENT VALUES OF TIME,RADIUS AND THETA;                                       115
FOR I←5     STEP 5     UNTIL 200    DO                                         116
    BEGIN K←K+1;                                                               117
    M←0;                                                                       118
    WRITE(HEAD2,I,1,TMX);                                                      119
    FOR J←50 STEP 50 UNTIL II  DO BEGIN                                        120
        M←M+1;                                                                 121
        R[K,M]←(X[J]×SQRT(2×I×AL));                                            122
        WRITE(HEAD1,I,R[K,M],MOIST[0,M]);END;                                  123
    END;                                                                       124
COMMENT DF IS 45 DEG;                                                          125
F[0]←1;F[1]←0.707;F[2]←0;                                                      126
F[3]←-F[1];F[4]←-1;                                                            127
M←0; K←0; CHK←TRUE;                                                            128
WRITE([PAGE]);                                                                 129
WRITE(EF);                                                                     130
P←P-1;SG←1.40;                                                                 131
FOR I←0 STEP 1 UNTIL 4 DO KP[I,P+1]←0;                                         132
FINAL:M←M+5;                                                                   133
      K←K+1;                                                                   134
      WRITE(T,M);                                                              135
      WRITE(GR,TMX,TMX,TMX,TMX,TMX);                                           136
FOR I←1 STEP 1 UNTIL P DO                                                      137
FOR J←0 STEP 1 UNTIL 4 DO                                                      138
BEGIN TEST←(KP[J,I-1]-KP[J,I])/(R[K,I]-R[K,I-1]);                              139
      THETA←(KP[J,I]-KP[J,I+1])/(R[K,I+1]-R[K,I]);                             140
      MOIST[J,I]←MOIST[J,I]+F[J]×(TEST+THETA)/(4×(@-1+D[J,I]));                141
      IF MOIST[J,I]<TMN  THEN MOIST[J,I]←TMN;                                  142
END;                                                                           143
FOR I←1 STEP 1 UNTIL P DO                                                      144
    WRITE(GR,FOR J←0 STEP 1 UNTIL 4 DO MOIST[J,I]);                            145
    WRITE(GR,TMN,TMN,TMN,TMN,TMN);                                             146
COMMENT INFILTRATION FOR UNIT THICKNESS OF MEDIUM;                             147
FOR I←1 STEP 1 UNTIL P+1 DO                                                    148
BEGIN TEMP←R[K,I]-R[K,I-1];                                                    149
      FOR J←0 STEP 1 UNTIL 3 DO                                                150
          TOTAL←(1.5707963×TEMP×(R[K,I]-TEMP/2)×                               151
          ((MOIST[J,I]+MOIST[J,I-1]+MOIST[J+1,I]+MOIST[J+1,I-1])/4-TMN))       152
          ×SG+TOTAL;END;                                                       153
      INFILT←TOTAL-TOTAL1;                                                     154
      IRATE←INFILT/5;                                                          155
      IVEL←IRATE/6.2831852;                                                    156
      TOTAL1←TOTAL;                                                            157
      WRITE(INF,M,TOTAL,IRATE,IVEL);                                           158
      TOTAL←0;                                                                 159
      IF K<40 THEN GO TO HARD;                                                 160
END;END.                                                                       161
```

B. FLOW CHART FOR ALGOL PROGRAM





ABBREVIATIONS

R-K	Runge-Kutta Method (3 steps only)
Apply Adams	Use Adams predictor formula once and Adams corrector formula desired times.
Lagrange	Lagrange Interpolation polynomial
ERRMAX	Maximum error
Y_F	Y_{FINAL}
TOL	TOLERANCE
INF	Infiltration rate

APPENDIX 2

THREE DIMENSIONAL DIFFUSION EQUATION

An investigation was made on the three dimensional diffusion equation. It is presented here for future reference and further investigations by interested readers. The general three dimensional diffusion equation in rectangular coordinates is

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(D \frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(D \frac{\partial \theta}{\partial z}\right) - \frac{\partial k}{\partial z} \qquad \text{(A2.1)}$$

or

$$- \frac{\partial k}{\partial z} + D\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right) + \left(\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial D}{\partial z} \cdot \frac{\partial \theta}{\partial z}\right) = \frac{\partial \theta}{\partial t} \qquad \text{(A2.2)}$$

Equation (A2.2) will be transformed to spherical coordinates. As shown in Figure A2.1,

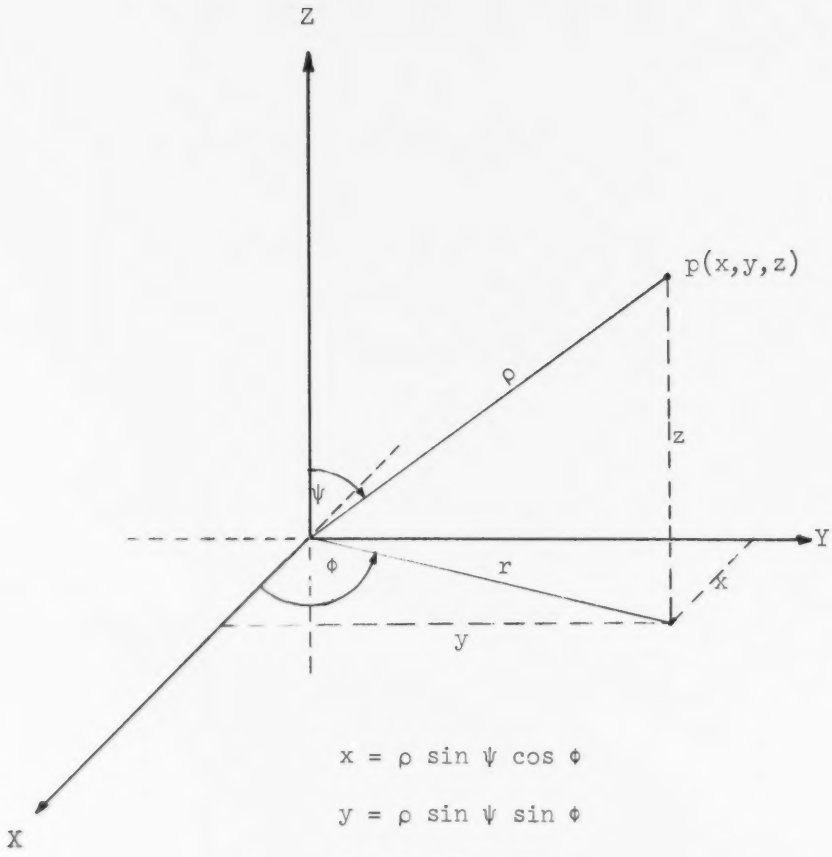
$$\rho = h_1(x,y,z) \quad ,$$

$$\psi = h_2(x,y,z) \quad ,$$

and

$$\Phi = h_3(x,y,z) \quad .$$

$$\therefore \quad \frac{\partial \theta}{\partial \rho} = \frac{\partial \theta}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial \theta}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial \theta}{\partial z} \cdot \frac{dz}{\partial \rho} \qquad \text{(A2.3.a)}$$



$$x = \rho \sin \psi \cos \phi$$

$$y = \rho \sin \psi \sin \phi$$

$$z = \rho \cos \psi$$

Fig. A2.1: Relation between Cartesian and Spherical Coordinates.

$$\frac{\partial\theta}{\partial\psi} = \frac{\partial\theta}{\partial x} \cdot \frac{\partial x}{\partial\psi} + \frac{\partial\theta}{\partial y} \cdot \frac{\partial y}{\partial\psi} + \frac{\partial\theta}{\partial z} \cdot \frac{\partial z}{\partial\psi} \qquad \text{(A2.3.b)}$$

$$\frac{\partial\theta}{\partial\phi} = \frac{\partial\theta}{\partial x} \cdot \frac{\partial x}{\partial\phi} + \frac{\partial\theta}{\partial y} \cdot \frac{\partial y}{\partial\phi} + \frac{\partial\theta}{\partial z} \cdot \frac{\partial z}{\partial\phi} \qquad \text{(A2.3.c)}$$

$$\left.\begin{aligned} \frac{\partial x}{\partial\rho} &= \frac{\partial(\rho \sin\psi \cos\phi)}{\partial\rho} = \sin\psi \cos\phi \\ \frac{\partial y}{\partial\rho} &= \frac{\partial(\rho \sin\psi \sin\phi)}{\partial\rho} = \sin\psi \sin\phi \\ \frac{\partial z}{\partial\rho} &= \frac{\partial(\rho \cos\psi)}{\partial\rho} = \cos\psi \end{aligned}\right\} \qquad \text{(A2.4.a)}$$

Similarly

$$\left.\begin{aligned} \frac{\partial x}{\partial\psi} &= \rho \cos\psi \cos\phi \\ \frac{\partial y}{\partial\psi} &= \rho \cos\psi \sin\phi \\ \frac{\partial z}{\partial\psi} &= -\rho \sin\psi \end{aligned}\right\} \qquad \text{(A2.4.b)}$$

and

$$\left.\begin{aligned} \frac{\partial x}{\partial\phi} &= -\rho \sin\psi \sin\phi \\ \frac{\partial y}{\partial\phi} &= +\rho \sin\psi \cos\phi \\ \frac{\partial z}{\partial\phi} &= 0 \end{aligned}\right\} \qquad \text{(A2.4.c)}$$

Subsituting Equations (A2.4.a), (A2.4.b) and (A2.4.c) into Equations (A2.3.a), (A2.3.b) and (A2.3.c) respectively and solving the three equations simultaneously, gives

$$\frac{\partial\theta}{\partial x} = \sin\psi\cos\phi\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho}\cos\psi\cos\phi\frac{\partial\theta}{\partial\psi} - \frac{\sin\phi}{\rho\sin\psi}\frac{\partial\theta}{\partial\phi} \qquad (A2.5)$$

$$\frac{\partial\theta}{\partial y} = \sin\psi\sin\phi\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho}\cos\psi\sin\phi\frac{\partial\theta}{\partial\psi} + \frac{\cos\phi}{\rho\sin\psi}\frac{\partial\theta}{\partial\phi} \qquad (A2.6)$$

$$\frac{\partial\theta}{\partial z} = \cos\psi\frac{\partial\theta}{\partial\rho} - \frac{1}{\rho}\sin\psi\frac{\partial\theta}{\partial\psi} \qquad (A2.7)$$

The left hand side of equation (A2.2) contains three terms. The transformations of the first and third terms will be shown here in detail, but the bracketed expression in the second term is the well-known Laplace equation, whose transformation in spherical coordinates can be found by the analogous technique as was done in polar coordinates for the two dimensional Laplace equation. Any advanced mathematics text will present the Laplace equation in spherical coordinates, and for the sake of brevity, it is written below without its full detail of derivation.

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\theta}{\partial\rho}\right) + \frac{1}{\rho^2\sin\psi}\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial\theta}{\partial\psi}\right) + \frac{1}{\rho^2\sin^2\psi}\left(\frac{\partial^2\theta}{\partial\phi^2}\right) \qquad (A2.8)$$

The first term of equation (A2.2) can be developed by replacing θ with k in Equation (A2.7) and changing sign,

$$-\frac{\partial k}{\partial z} = -\left[\cos\psi\frac{\partial k}{\partial\rho} + \frac{1}{\rho}\sin\psi\frac{\partial k}{\partial\psi}\right] \qquad (A2.9)$$

For finding the third term of Equation (A2.2), $\frac{\partial D}{\partial x}$, $\frac{\partial D}{\partial y}$ and $\frac{\partial D}{\partial z}$ are needed, which can be found by replacing θ in Equations (A2.5), (A2.6) and (A2.7) with D.

$$\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} = \sin^2\psi \cos^2\phi \ \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} + \frac{1}{\rho^2} \cos^2\psi \cos^2\phi \ \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \frac{\sin^2\phi}{\rho^2 \sin^2\psi} \ \frac{\partial \theta}{\partial \phi} \cdot \frac{\partial D}{\partial \phi} + \sin\psi \cos\psi \cos^2\phi \cdot \frac{1}{\rho} \cdot \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \sin\psi \cos\psi \cos^2\phi \cdot \frac{1}{\rho} \cdot \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \psi} - \frac{1}{\rho} \sin\phi \cos\phi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \phi}$$

$$- \frac{1}{\rho} \sin\phi \cos\phi \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \phi} - \frac{1}{\rho^2} \frac{\cos\psi}{\sin\psi} \sin\phi \cos\phi \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \phi}$$

$$- \frac{1}{\rho^2} \frac{\cos\psi}{\sin\psi} \sin\phi \cos\phi \frac{\partial D}{\partial \psi} \cdot \frac{\partial \theta}{\partial \phi} \qquad \text{(A2.10)}$$

$$\frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} = \sin^2\psi \sin^2\phi \ \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} + \frac{1}{\rho^2} \cos^2\psi \sin^2\phi \ \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \frac{\cos^2\phi}{\rho^2 \sin^2\psi} \ \frac{\partial \theta}{\partial \phi} \cdot \frac{\partial D}{\partial \phi} + \sin\psi \cos\psi \sin^2\phi \cdot \frac{1}{\rho} \ \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \sin\psi \cos\psi \sin^2\phi \cdot \frac{1}{\rho} \cdot \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \psi} + \frac{1}{\rho} \sin\phi \cos\phi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \phi}$$

$$+ \frac{1}{\rho} \sin\phi \cos\phi \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \phi} + \frac{1}{\rho^2} \frac{\cos\psi}{\sin\psi} \sin\phi \cos\phi \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \phi}$$

$$+ \frac{1}{\rho^2} \frac{\cos\psi}{\sin\psi} \sin\phi \cos\phi \frac{\partial D}{\partial \psi} \cdot \frac{\partial \theta}{\partial \phi} \qquad \text{(A2.11)}$$

Adding Equations (A2.10) and (A2.11) and cancelling common terms with opposite signs,

$$\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} = \sin^2\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} (\cos^2\phi + \sin^2\phi)$$

$$+ \frac{1}{\rho^2} \cos^2\psi \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi} (\cos^2\phi + \sin^2\phi)$$

$$+ \frac{1}{\rho^2 \sin^2\psi} \frac{\partial \theta}{\partial \phi} \cdot \frac{\partial D}{\partial \phi} (\sin^2\phi + \cos^2\phi)$$

$$+ \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \psi} (\cos^2\phi + \sin^2\phi)$$

$$+ \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \psi} (\cos^2\phi + \sin^2\phi)$$

or $$\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} = \sin^2\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} + \frac{1}{\rho^2} \cos^2\psi \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \frac{1}{\rho^2 \sin^2\psi} \frac{\partial \theta}{\partial \phi} \cdot \frac{\partial D}{\partial \phi} + \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \psi}$$

$$+ \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \psi} \qquad \text{(A2.12)}$$

Now

$$\frac{\partial D}{\partial z} \cdot \frac{\partial \theta}{\partial z} = \cos^2\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} + \frac{1}{\rho^2} \sin^2\psi \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi} - \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \psi}$$

$$- \frac{1}{\rho} \sin\psi \cos\psi \frac{\partial D}{\partial \rho} \cdot \frac{\partial \theta}{\partial \psi} \qquad \text{(A2.13)}$$

Adding again Equations (A2.12) and (A2.13) and cancelling common terms with opposite signs,

$$\frac{\partial D}{\partial x} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial D}{\partial y} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial D}{\partial z} \cdot \frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial \rho} \cdot \frac{\partial D}{\partial \rho} (\sin^2\psi + \cos^2\psi)$$

$$+ \frac{1}{\rho^2} \frac{\partial \theta}{\partial \psi} \cdot \frac{\partial D}{\partial \psi} (\sin^2\psi + \cos^2\psi)$$

$$+ \frac{1}{\rho^2 \sin^2\psi} \frac{\partial \theta}{\partial \phi} \cdot \frac{\partial D}{\partial \phi}$$

or $$\frac{\partial D}{\partial x}\cdot\frac{\partial \theta}{\partial x}+\frac{\partial D}{\partial y}\cdot\frac{\partial \theta}{\partial y}+\frac{\partial D}{\partial z}\cdot\frac{\partial \theta}{\partial z}=\frac{\partial \theta}{\partial \rho}\cdot\frac{\partial D}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial \theta}{\partial \psi}\cdot\frac{\partial D}{\partial \psi}$$

$$+\frac{1}{\rho^2\sin^2\psi}\frac{\partial \theta}{\partial \phi}\cdot\frac{\partial D}{\partial \phi} \qquad \text{(A2.14)}$$

Finally adding Equations (A2.8) multiplied by D, (A2.9) and (A2.14), gives the left side of Equation (A2.2). The diffusion equation in spherical coordinates thus becomes

$$-\left[\cos\psi\frac{\partial k}{\partial \rho}+\frac{1}{\rho}\sin\psi\frac{\partial k}{\partial \psi}\right]+D\left[\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial \theta}{\partial \rho}\right)+\frac{1}{\rho^2\sin\psi}\frac{\partial}{\partial \psi}\left(\sin\psi\frac{\partial \theta}{\partial \psi}\right)\right.$$

$$\left.+\frac{1}{\rho^2\sin^2\psi}\left(\frac{\partial^2\theta}{\partial \phi^2}\right)\right]+\left[\frac{\partial \theta}{\partial \rho}\cdot\frac{\partial D}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial \theta}{\partial \psi}\cdot\frac{\partial D}{\partial \psi}+\frac{1}{\rho^2\sin^2\psi}\frac{\partial \theta}{\partial \phi}\cdot\frac{\partial D}{\partial \phi}\right]$$

$$=\frac{\partial \theta}{\partial t} \qquad \text{(A2.15)}$$

Noting that

$$\frac{D}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial \theta}{\partial \rho}\right)+\frac{\partial \theta}{\partial \rho}\cdot\frac{\partial D}{\partial \rho}=\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(D\rho^2\frac{\partial \theta}{\partial \rho}\right)\ ,$$

$$\frac{D}{\rho^2\sin\psi}\frac{\partial}{\partial \psi}\left(\sin\psi\frac{\partial \theta}{\partial \psi}\right)+\frac{1}{\rho^2}\frac{\partial \theta}{\partial \psi}\cdot\frac{\partial D}{\partial \psi}=\frac{1}{\rho^2\sin\psi}\frac{\partial}{\partial \psi}\left(D\sin\psi\frac{\partial \theta}{\partial \psi}\right),\ \text{and}$$

$$\frac{D}{\rho^2\sin^2\psi}\left(\frac{\partial^2\theta}{\partial \phi^2}\right)+\frac{1}{\rho^2\sin^2\psi}\frac{\partial \theta}{\partial \phi}\cdot\frac{\partial D}{\partial \phi}=\frac{1}{\rho^2\sin^2\psi}\frac{\partial}{\partial \phi}\left(D\frac{\partial \theta}{\partial \phi}\right)\ ,$$

the three dimensional diffusion equation in spherical coordinates can be written in a more compact form as

$$\frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(D\rho^2\frac{\partial\theta}{\partial\rho}\right)+\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}\left(D\sin\psi\frac{\partial\theta}{\partial\psi}\right)+\frac{1}{\sin^2\psi}\frac{\partial}{\partial\phi}\left(D\frac{\partial\theta}{\partial\phi}\right)\right]$$

$$-\left[\cos\psi\frac{\partial k}{\partial\rho}+\frac{1}{\rho}\sin\psi\frac{\partial k}{\partial\psi}\right]=\frac{\partial\theta}{\partial t} \qquad \text{(A2.16)}$$

The three dimensional diffusion equation in the form of Equation (A2.16) is especially suited for the unsteady and unsaturated flow into soils from a spherical source. After neglecting gravity, the flow becomes independent of ϕ and ψ and Equation (A2.16) takes a simple form,

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(D\rho^2\frac{\partial\theta}{\partial\rho}\right)=\frac{\partial\theta}{\partial t} \qquad \text{(A2.17)}$$

Expanding Equation (A2.17), gives

$$\frac{\partial D}{\partial\rho}\cdot\frac{\partial\theta}{\partial\rho}+\frac{D}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\theta}{\partial\rho}\right)=\frac{\partial\theta}{\partial t} \qquad \text{(A2.18.a)}$$

with boundary conditions

$$\left.\begin{aligned}\rho=\rho_o\ ,\quad \theta=\theta_{sat} \qquad t>0\\ \rho=\infty\ ,\quad \theta=\theta_n \qquad t>0\end{aligned}\right\} \qquad \text{(A2.18.b)}$$

Similar to Equation (4.14) for the case of two dimensional flow, Equation (A2.18) can be transformed to a nonlinear, second-order, ordinary differential equation. For this, using the following transformations and the relation for diffusivity

$$x = \frac{\rho}{\sqrt{2\alpha t}}$$

$$y = e^{\beta\theta}$$

or $$\theta = \frac{1}{\beta} \ell n\ y$$

and $$D = \alpha e^{\beta\theta} \quad ,$$

the following derivatives are computed:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial t} = \frac{\partial}{\partial x} \cdot \left(-\frac{\rho\alpha}{\sqrt{2\alpha t^3}}\right)$$

$$\frac{\partial}{\partial \rho} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \rho} = \frac{\partial}{\partial x} \cdot \frac{1}{\sqrt{2\alpha t}}$$

Substituting back into Equation (A2.18)

$$\frac{1}{2\alpha t} \cdot \frac{\partial(\alpha e^{\beta\theta})}{\partial x} \cdot \frac{\partial(\frac{1}{\beta} \ell n\ y)}{\partial x} + \frac{\alpha e^{\beta\theta}}{x^2(2\alpha t)} \cdot \frac{1}{\sqrt{2\alpha t}} \cdot \frac{\partial}{\partial x} \left(x^2(2\alpha t) \cdot \frac{1}{\sqrt{2\alpha t}} \cdot \frac{\partial(\frac{1}{\beta} \ell n\ y)}{\partial x}\right)$$

$$= -\frac{\partial(\frac{1}{\beta} \ell n\ y)}{\partial x} \cdot \frac{\alpha x \sqrt{2\alpha t}}{(2\alpha t)^{3/2}}$$

or $$\sqrt{2\alpha t} \cdot \alpha\beta e^{\beta\theta} \cdot \frac{\partial(\frac{1}{\beta} \ell n\ y)}{\partial x} \cdot \frac{\partial(\ell n\ y)}{\partial x} + \frac{\alpha e^{\beta\theta}}{x^2} \cdot \frac{\partial}{\partial x} \left(x^2 \cdot \sqrt{2\alpha t} \cdot \frac{\partial(\ell n\ y)}{\partial x}\right)$$

$$= -\frac{\partial(\ell n\ y)}{\partial x} \cdot \alpha x \sqrt{2\alpha t}$$

$$\text{or} \quad \sqrt{2\alpha t}\,\alpha y \cdot \frac{y'}{y} \cdot \frac{y'}{y} + \frac{\alpha y}{x^2} \cdot 2x\left(\sqrt{2\alpha t} \cdot \frac{y'}{y}\right) + \frac{\alpha y}{x^2} \cdot x^2 \sqrt{2\alpha t} \cdot \frac{y''}{y}$$

$$- \frac{\alpha y}{x^2} \cdot x^2 \sqrt{2\alpha t} \cdot \frac{y'^2}{y^2} = - \frac{y'}{y} \cdot \alpha x \sqrt{2\alpha t}$$

$$\text{or} \quad \frac{y'^2}{y} + \frac{2y'}{x} + y'' - \frac{y'^2}{y} = - \frac{y'}{y}\, x$$

$$\therefore \qquad y'' = - y'\left(\frac{2}{x} + \frac{x}{y}\right) \tag{A2.19.a}$$

subject to conditions for $t > 0$,

$$x = x_o = \frac{\rho_o}{\sqrt{2\alpha t_o}} , \quad y = e^{\beta\theta_{sat}} \tag{A2.19.b}$$

$$x = \infty \qquad , \quad y = e^{\beta\theta_n} \tag{A2.19.b}$$

Equation (A2.19) is for the three dimensional case. This equation is similar to the two dimensional Equation (6.11). By arguments similar to those used for the transformed two dimensional equation, Equation (A2.19) can be shown to possess the asymptotic behavior at a distance ρ_1 at a finite time t_1, and the boundary condition at infinity can be replaced in this case also. The method developed for the two dimensional problem could be applied with proper modification to the three dimensional problem because of the similarity in the nature of the equations and their solutions. This is a good area for future research.

BIBLIOGRAPHY

1. Adrian, D. D., The Influence of Capillarity on the Flow of Air and Water in a Porous Medium. Ph.D. dissertation. Published as Civil Engineering Department Technical Report No. 38, Stanford University, (1964).

2. Ashcroft, G., D. D. Marsh, D. D. Evans and L. Boersma, Numerical Methods for Solving the Diffusion Equation: I. Horizontal Flow in Semi-Infinite Media. Soil Sci. Soc. Amer. Proc., 26: 522-525 (1962).

3. Baver, L. D., Soil Physics, John Wiley and Sons, Inc., New York, Third Edition: 81-303 (1959).

4. Bodman, G. B., The Variability of the Permeability Constant at Low Hydraulic Gradients During Saturated Water Flow in Soils. Soil Sci. Soc. Amer. Proc. 2: 45-53 (1938).

5. Bodman, G. B., and Colman, E. A., Moisture and Energy Conditions during Downward Entry of Water into Soils, Soil Sci. Soc. Amer. Proc. 8: 116-122 (1944).

6. Buckingham, E., Studies on the Movement of Soil Moisture, U. S. Dept. Agr. Bur. Soils, Bul. 38: 1-61 (1907).

7. Burroughs Corporation, Extended Algol Reference Manual for the Burroughs B 5000, Bul. 5000-21012, Detroit, Michigan (1962).

8. Burroughs Corporation, Miscellaneous Facilities of Extended Algol for the Burroughs B 5000, Bul. 5000-21013, Detroit, Michigan (1963).

9. Childs, E. C., Recent Advances in the Study of Water Movement in Unsaturated Soil, Trans. Intern. Congr. Soil Sci., 6th Congr. 39: 265-274 (1956).

10. Childs, E. C. and N. C. George, The Permeability of Porous Materials, Proc. Roy. Soc. (London), 210A: 392-399 (1950).

11. Davidson, J. M., J. W. Biggar and D. R. Nielsen, Gamma-Radiation Attenuation for Measuring Bulk Density and Transient Water Flow in Porous Materials, Jour. Geophys. Res., Vol. 68, No. 16: 4777-4783 (1963).

12. Gardner, W. R., Some Steady-State Solutions of the Unsaturated Moisture Flow Equation with Application to Evaporation from a Water Table. Soil Sci. Vol. 85, No. 4: 228-232 (1958).

13. Gardner, W. R. and D. I. Hillel, The Relation of External Evaporation Conditions to the Drying of Soils. Journ. of Geophys. Res. 67: 4319-4325 (1962).

14. Gardner, W. R. and M. S. Mayhugh, Solutions and Tests of the Diffusion Equation for the Movement of Water in Soil, Soil Sci. Soc. Amer. Proc., 22: 197-201 (1958).

15. Hanks, R. J. and S. A. Bowers, Numerical Solution of the Moisture Flow Equation for Infiltration into Layered Soils, Soil Sci. Soc. Amer. Proc., pp. 530-534 (1962).

16. Harr, M. E., Ground Water and Seepage, McGraw-Hill Book Company, New York (1962).

17. Hildebrand, F. B., Advanced Calculus for Applications, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1962).

18. Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

19. Hildebrand, F. B., Introduction to Numerical Analysis, McGraw-Hill Book Company, Inc., New York (1956).

20. Klute, A., Some Theoretical Aspects of the Flow of Water in Unsaturated Soils, Soil Sci. Soc. Amer. Proc., 16: 144-148 (1951).

21. Klute, A., A Numerical Method for Solving the Flow Equation for Water in Unsaturated Materials, Soil Sci. Soc. Amer. Proc., 73: 105-116 (1952).

22. Lamb, Sir Horace, Hydrodynamics, sixth Edition, Dover Publications, Inc., New York (1932).

23. Kutta, W., Beitrag zur näherungsweisen Integration totaler Differentialgleichungen, Z. Math. Physics, 46, 435-453 (1901).

24. Linsley, Ray K. and Joseph B. Franzini, Elements of Hydraulic Engineering, McGraw-Hill Book Company, Inc., New York, 486-523 (1955).

25. Linsley, Ray K., Max A. Kohler and Joseph L. H. Paulhus, Hydrology for Engineers, McGraw-Hill Book Company, Inc., New York, 122-148 (1959).

26. McCracken, Daniel D., A Guide to Algol Programming, John Wiley and Sons, Inc., New York (1962).

27. Moore, R. E., Water Conduction from Shallow Water Tables, Hilgardia, 12: 383-426 (1939).

28. Nelson, R. W., Steady Darcian Transport of Fluids in Hetrogeneous Partially Saturated Porous Media, Part 1, Mathematical and Numerical Formulation, U. S. Atomic Energy Commission and General Electric Company, Richland, Washington, HW 72335, TID-4500, 17th Edition (1962).

29. Ohmstede, William D., Numerical Solution of Nonsteady Soil Moisture Transfer, I., Signal Corps Meteorological Program, DA Task 3A99-27-005-08 (1957).

30. Philip, J. R., Numerical Solution of Equations of the Diffusion Type with Diffusivity Concentration-Dependent, Trans. Faraday Soc., 51: 885-892 (1955).

31. Philip, J. R., The Theory of Infiltration: 1. The Infiltration Equation and Its Solution. Soil Sci., 83: 345-357 (1957).

32. Philip, J. R., The Theory of Infiltration: 2. The Profile of Infinity, Soil Sci., 83: 435-448 (1957).

33. Philip, J. R., The Theory of Infiltration: 3. Moisture Profiles and Relation to Experiment, Soil Sci., 84: 163-178 (1957).

34. Reisenauer, A. E. and R. W. Nelson, Some Influences of Soils and Water Table Depths on Seepage from Lined and Unlined Canals, General Electric Company, Richland, Washington (1963).

35. Reisenauer, A. E., R. W. Nelson and C. N. Knudsen, Steady Darcian Transport of Fluids in Hetrogeneous Partially Saturated Porous

Media, Part 2, The Computer Program, HW-72335 PT2, TID-4500, 26th Ed., U.S. Atomic Energy Commission and General Electric Company, Richland, Washington (1963).

36. Remson, Irwin, Ronald L. Drake, Samuel S. McNeary and Edward M. Wallo, Vertical Drainage of an Unsaturated Soil, Journ. Hydr. Div., Proc. Amer. Soc. Civ. Engr., pp. 55-74 (1965).

37. Richards, L. A., Chairman, Report of the Subcommittee on Permeability and Infiltration, Committee on Terminology, Soil Sci. Soc. Amer., Soil Sci. Proc., pp. 85-88 (1951).

38. Richards, L. A., Capillarity Conduction of Liquids in Porous Mediums, Physics, 1: 318-333 (1931).

39. Richards, L. A., Retention and Transmission of Water in Soil, U. S. Salinity Lab., Riverside, Calif., Yearbook of Agr., Separate No. 2580: 144-151 (1955).

40. Richards, L. A. and S. J. Richards, Soil Moisture, U. S. Salinity Lab., Riverside, Calif., Yearbook of Agr., Separate No. 2788: 49-60 (1957).

41. Rubin, J. and R. Steinhardt, Soil-Water Relations During Rain Infiltration: I. Theory, Soil Sci. Soc. Amer. Proc., 27: 246-251 (1963).

42. Ruehrwein, R. A. and D. W. Ward, Mechanism of Clay Aggregation by Polyelectrolytes, Soil Sci., 73: 485-491 (1952).

43. Runge, C., Über die numerische Auflösung von Differentialgleichungen, Math. Ann., 46, 167-178 (1895).

44. Russell, M. B. and A. Klute, Movement of Water in Soils, Journ. Amer. Soc. Agr. Engr., Vol. 35, No. 11: 808-810 (1954).

45. Schleusener, Richard A. and A. T. Corey, The Role of Hysterisis in Reducing Evaporation from Soils in Contact with a Water Table, Journ. Geophys. Res., Vol. 64, No. 4: 469-475 (1959).

46. Schofield, R. K., The pF of the Water in Soil, Trans. Third Inter. Congr. Soil Sci., 2: 37-48 (1935).

47. Scott, E. J. and R. J. Hanks, Solution of theOne-Dimensional Diffusion Equation for Exponential and Linear Diffusivity Functions by Power Series Applied to Moisture Flow in Soils, Soil Sci., Vol. 94, No. 5: 314-321 (1962).

48. Scott, E. J., R. J. Hanks, D. B. Peters and A. Klute, Power Series Solutions of One-Dimensional Flow Equation for Exponential and Linear Diffusivity Functions, U. S. Dept. of Agr., ARS 41-64 (1962).

49. Todd, David K., Ground Water Hydrology, John Wiley and Sons, New York (1959).

50. Van Bavel, C. H. M., P. R. Nixon and V. L. Hauser, Soil Moisture Measurement with the Neutron Method, U. S. Dept. of Agr., ARS-41-70 (1963).

51. Wang, F. C., An Approach to the Solution of Unsteady and Unsaturated Flow Problems in Soils, Ph.D. dissertation. Published as Civil Engineering Department Technical Report No. 19, Stanford University (1963).

52. Wang, F. C., N. A. Hassan and J. B. Franzini, A Method of Analyzing Unsteady, Unsaturated Flow in Soils, Journ. Geophys. Res., Vol. 69, No. 12: 2569-2577 (1964).